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# STATICS

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# STATICS

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# PREFACE

It is now generally agreed that the school course of the non-specialist in Mathematics is incomplete unless it includes an elementary treatment of Mechanics.

This book aims at providing in Part I. an inexpensive text-book suitable for such boys. The complete work in two parts will form an introductory course suitable for those reading for the Army and also for Engineering students.

In many cases it is impossible to provide accommodation for the class to perform the experiments, but in such circumstances it is highly desirable that the experiments in the text should be carried out by the master before the class. The readings can be given by the master, and the class should write an account of the experiments and make the necessary deductions.

For the private student the order of the work is recommended, but a practical difficulty arises in class teaching owing to the promotion of beginners to join those who commenced the subject the previous term. This can be met by arranging that one term is given to a course on Moments and Machines (Chapters I., II., V.), while the next term is devoted to Vectors (Chapters III. and IV.).

Part II. extends the application of the principles discussed in Part I. As far as possible each part of the subject is treated both graphically and analytically, and it is hoped that the book will meet the needs of Woolwich candidates, engineering students, and those reading for scholarships in Mathematics and Mechanical Sciences. It includes an elementary treatment of frameworks, bending moments, and forces in three dimensions, and will serve as an introduction to the more advanced treatises on Theory of Structures and Strength of Materials. Virtual work has been omitted as being more suitably dealt with by students familiar with the Calculus.

The examples are of a practical type, and have been selected to illustrate the principles of Statics without involving elaborate trigonometrical and algebraic manipulation. Their value is much enhanced by the inclusion of questions set at Cambridge University and in Army Examinations, by kind permission of the Syndics of the University Press and the Controller of His Majesty's Stationery Office.

My warmest thanks are due to my colleagues Dr. W. P. Milne and Mr. H. Clissold, also to Mr. T. S. Usherwood of Christ's Hospital, for much valuable assistance and advice.

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In this edition will be found, in the form of an Appendix, a collection of examples which involve a knowledge of trigonometrical manipulation to meet the needs of students preparing for the London B.Sc. Examination.

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***NOTE.*** *This book is issued Complete and also Part I separately. The following is the Table of Contents of the whole book.*

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ANSWERS.

# INTRODUCTORY.

MECHANICS is the science which treats of the conditions of rest or of motion of bodies.

Statics is the branch of mechanics which treats of bodies in a state of rest, the forces acting on the body are then said to be in equilibrium.

Dynamics deals with bodies in motion.

**Force.**

Experience has made us familiar with the idea of a Force.

If a body is at rest, it can only be made to move by the action of a force. We may produce the force, for instance, by pushing the body, or by pulling it with a string. If the body is resting on a rough surface, the force which we exert may be insufficient to produce motion; in this case there is another force, called friction, acting on the body in the opposite direction to which we are pushing or pulling. If the support is removed, the body will fall vertically downwards. The force causing this motion is the attraction of the earth, and is called the weight of the body. Before the support was removed, the body was also acted upon by the upward pressure of the table.

If a body is in motion, it can only be stopped, or caused to deviate from motion in a straight line, by the action of

a force. A stone thrown along the surface of a sheet of ice will eventually be brought to rest by friction and the resistance of the air.

It is important to notice that, although the weight of a body is constant at a given place, it is not always the same at different places. Its weight is due to the pull on it produced by the attraction of the particles of the earth. This force acts very nearly towards the centre of the earth. The weight of a body is slightly less at the Equator than at the poles, owing to the fact that the earth is not a perfect sphere. Isaac Newton (1642-1727) showed that this attractive force is a property of all matter. A body weighed by a spring balance on the moon would only produce about one-fifth of the extension of the spring which it would produce on the earth's surface, since the attraction between the body and the moon is much less than the attraction between the body and the earth.

It will be noticed that the word 'weight' is used in two different senses. It is the name given to the metal bodies which we put into a scale pan for the purpose of 'weighing,' and it also denotes the force with which bodies are attracted to the earth.

**The Dynamometer.**

This instrument, which is merely a spiral spring, affords a convenient means of measuring forces.

Two forces may be said to be equal, if each of them, when applied in turn to a spiral spring, causes the same amount of extension. (We assume that when the first force is removed the spring returns to its original condition.)

This force may be produced by the weight of a body attached to the spring.

If two bodies, each of which causes the same extension of the spring, are together hung from the spring, we find that the extension is doubled, and we can show in a similar way that the extension of the spring is proportional to the force which is stretching it.

A graduated spiral spring is called a Dynamometer or a Spring Balance.

If we pull the end of the spiral spring until it is stretched to the graduation marked 300 grams, we can measure the pull we are exerting as being equal to the weight of 300 grams.

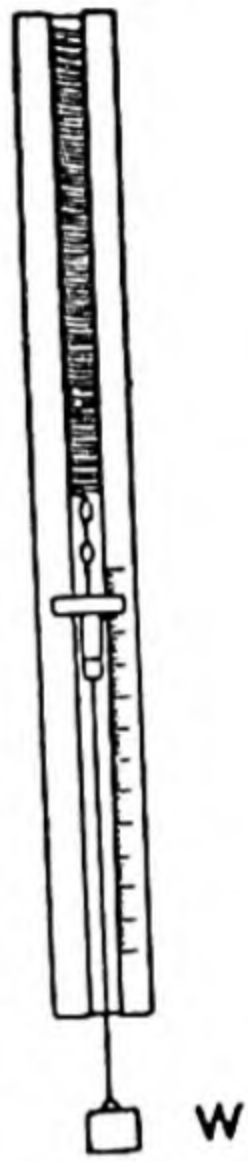


FIG. 1.

When studying Dynamics, the student will find a more scientific method of measuring forces, but the method given above will suffice for the present.[1]

If a weight W is attached to the dynamometer by a piece of string, we naturally say that the weight is producing the extension of the spring, but it is important to notice that the force acting directly on the dynamometer is produced by the string which is fastened to it. The force produced by a stretched string is called a Tension (T), and it is this force T which is acting on the dynamometer.

If we consider the equilibrium of the weight, we note that it is acted upon by two forces, its weight (W), downwards, and the tension of the string (T), upwards. Since the body is at rest, we deduce that

$T = W$.

FIG. 2.

It will be noted that we have neglected the weight of the string. The force acting downwards on the

[1] A convenient dynamometer measuring up to 500 grams is the "Rintoul Dynamometer" supplied by Messrs. Harris, Birmingham. The Salter spring balance is useful for larger forces.

dynamometer is really $T+w$ (where $w$ is the weight of the string).

Now, let us consider the equilibrium of a small portion AB of the string.

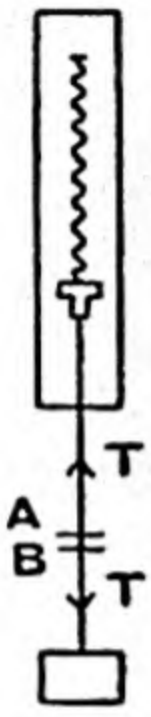

FIG. 3.

It will be acted upon by the tension in the lower part of the string below B, downwards, and by the tension in the upper part above A, upwards. If we neglect the weight of the portion of string AB, it follows that these two tensions are equal and that the tension throughout the string is the same.

Tie a knot A in the middle of a piece of string.

Fasten one end B of the string to a staple in the wall, and the other end to a dynamometer. Pull the string horizontally until the dynamometer registers 300 grams.

What is the pull on the staple? In which direction?

What is the pull on the dynamometer? In which direction?

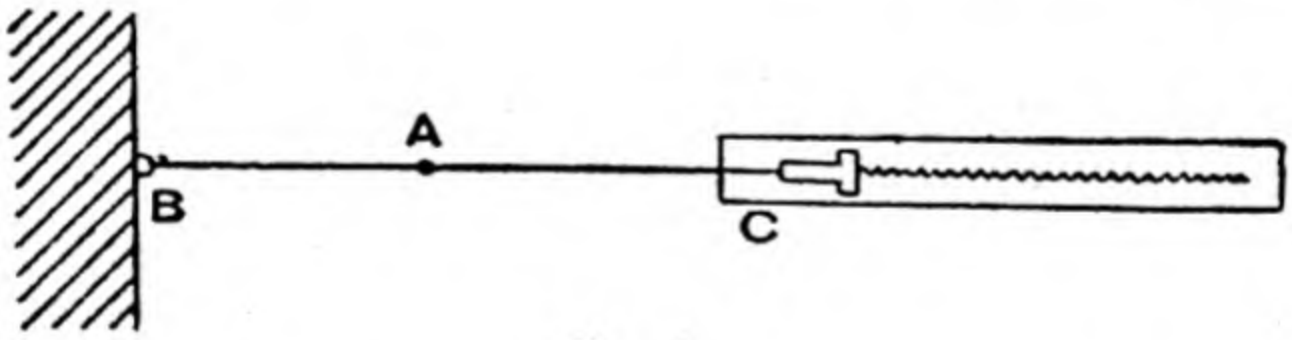

FIG. 4.

Why is the knot at A in equilibrium? State the forces acting on it, and give their direction.

It will be necessary, when using a dynamometer which is at all sensitive, to make adjustments for the angle at which it is inclined to the vertical. If the zero reading is correct when the dynamometer is vertical, the pointer will not read zero if the instrument is inclined at an angle to the vertical, and it must be readjusted.

Adjust the dynamometer to read correctly when inclined at 45° to the vertical, and then fasten to it a cord which passes over a pulley and supports a weight W.

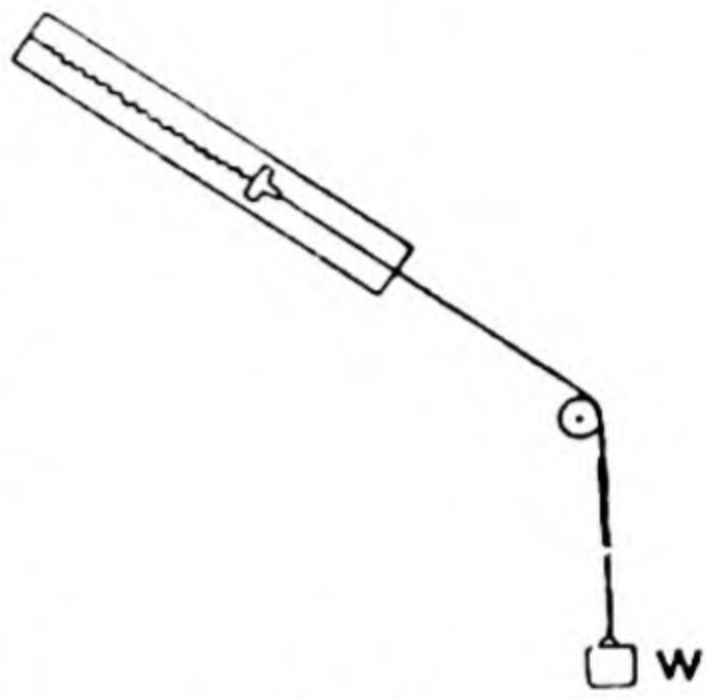


FIG. 5.

It will be found that the pull registered by the dynamometer is approximately equal to W, and becomes more nearly equal to W the more we reduce the friction of the pulley.

If we take two readings, one when W is slowly raised, and one when W is slowly lowered, the average of the two readings will give the pull, if there were no friction; this will be shown in Chapter II.

It appears, therefore, that the pulley, if frictionless, will not alter the tension of a string which passes over it, but merely changes its direction.

By this method we can find the correction to be applied to the dynamometer readings, if the dynamometer is adjusted to read correctly when vertical, and is then used upside down, so that the two portions of the string are parallel.

If a string is placed over two pulleys, as in the figure,

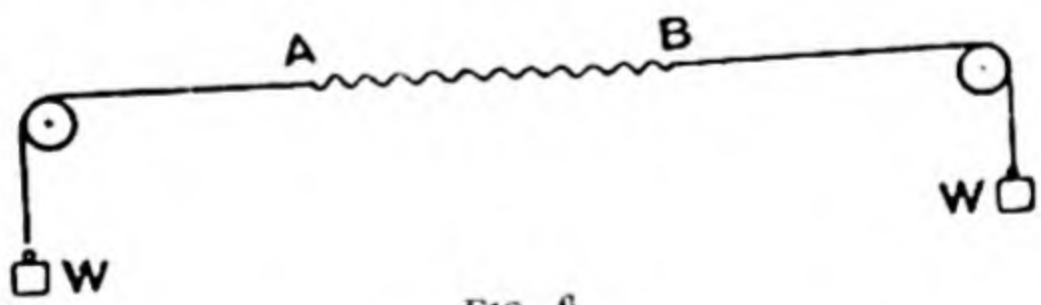


FIG. 6.

and a weight W is fastened to each end, what will be the tension in the string?

Insert a light spring AB between the pulleys, and measure its length when stretched. Then hang it up at one end A and find what weight is required to stretch it to the same extent.

**Forces in Equilibrium.**

One of the forces acting on a body is its weight. In order that the body may be in equilibrium, it must be acted upon by at least one other force.

Fasten two strings to two points in a stiff piece of cardboard, and attach the strings to two dynamometers.

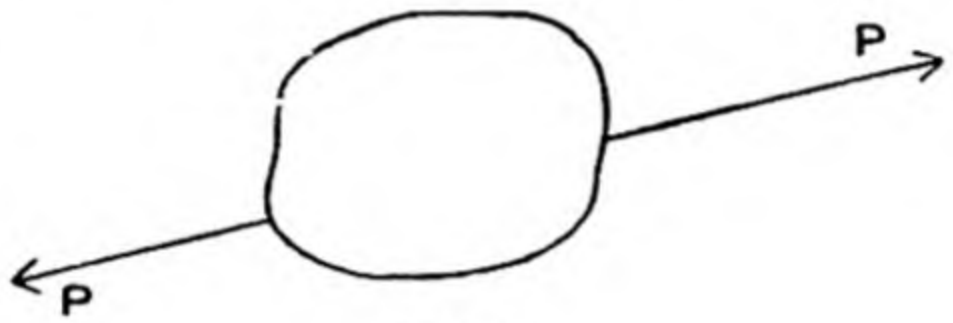


FIG. 7.

Place the cardboard on a table and pull the dynamometers, keeping them flat on the table. It will be found that the cardboard will only be in equilibrium when the two strings are in the same straight line, pulling in opposite directions, and the dynamometer readings are equal.

When these conditions are satisfied, these two forces do not disturb the equilibrium of the cardboard. It will be noticed that in this case the cardboard is also acted upon by its weight and by the pressure of the table, and is in equilibrium under the action of these forces before the introduction of the two additional forces.

We conclude that two forces acting on a body are in equilibrium when they are equal and opposite, and act in the same straight line. This result was assumed on p. 4,

when discussing the equilibrium of a weight supported by a string.

**Centre of Gravity.**

Balance a rectangular piece of cardboard on the point of a pencil. Each particle of the cardboard is attracted towards the centre of the earth, which is so far distant that

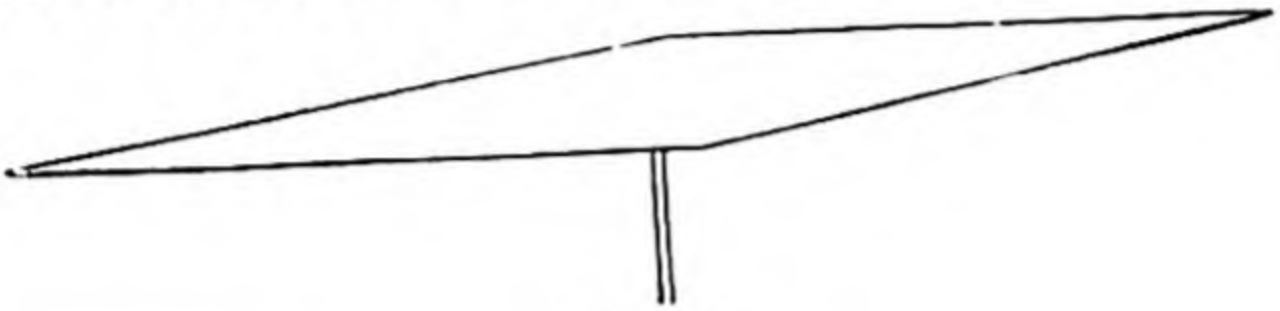

FIG. 8.

the forces may be considered parallel. All these forces are balanced by one upward vertical force, so that we can consider them to be equivalent to one force equal to the weight of the whole cardboard, acting vertically downwards at its middle point.

If we hang the cardboard up at one of its angular points, we shall find that the cardboard will hang with a diagonal vertical; mark this vertical line on the cardboard.

If we hang it from any point whatever, we shall find that the vertical line through this point always passes through the mid point of the rectangle.

The point in a body through

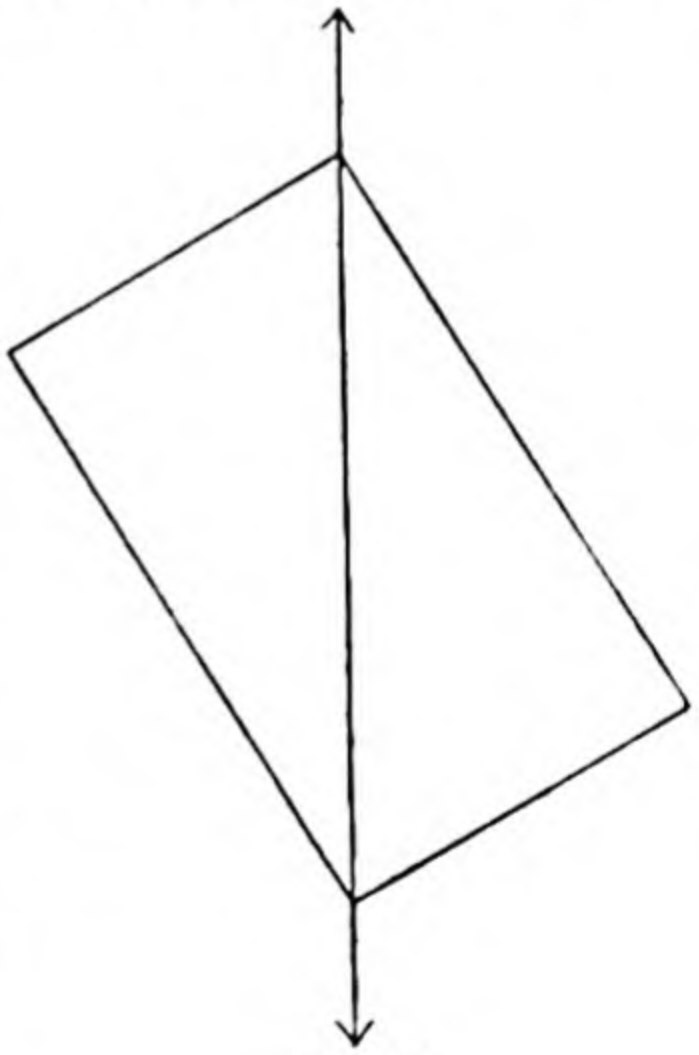

FIG. 9.

which the line of action of the weight always passes, whatever the position of the body, is called the Centre of Gravity of the body.

If we try the same experiment with a cubical block of wood, we shall find that this point is situated inside the wood where the diagonals of the cube intersect.

**Rigidity.**

If we fasten two equal weights to the extremities of a light thin rod, we shall be able to support the rod by an

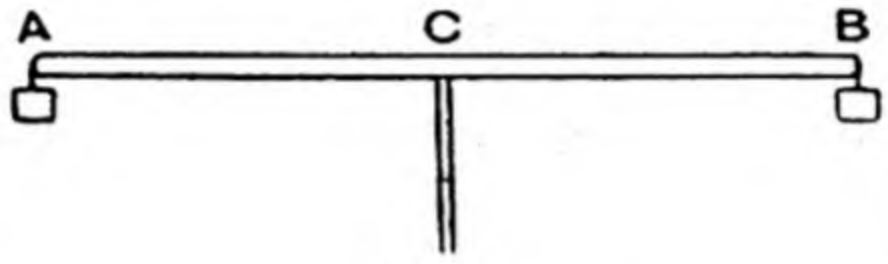


FIG. 10.

upward force at the mid point C, but there will be a tendency for the rod to break at C. If, however, we placed both weights at C there would be no such tendency.

It is clear, therefore, that if the distortion of the rod is to be considered, the combined effect of the weights at A and B is not the same as one single force acting at C.

A body which is not distorted when a force is applied to it is called a *Rigid body*.

Bodies made of steel or iron are not perfectly rigid, although it is difficult to detect any distortion if the forces acting on such bodies are not very great.

We shall assume that the bodies with which we have to deal are for all practical purposes rigid bodies, and that instead of considering the effect of the weight of each particle of the body separately, we may consider only their

combined effect, equal to the weight of the whole body, acting at the centre of gravity of the body.

**Pressure.**

If we place a rectangular block of wood (ABCD, A′B′C′D′) on a smooth horizontal table, it will be supported by the upward pressure of the table on the lower face A′B′C′D′ of the block.

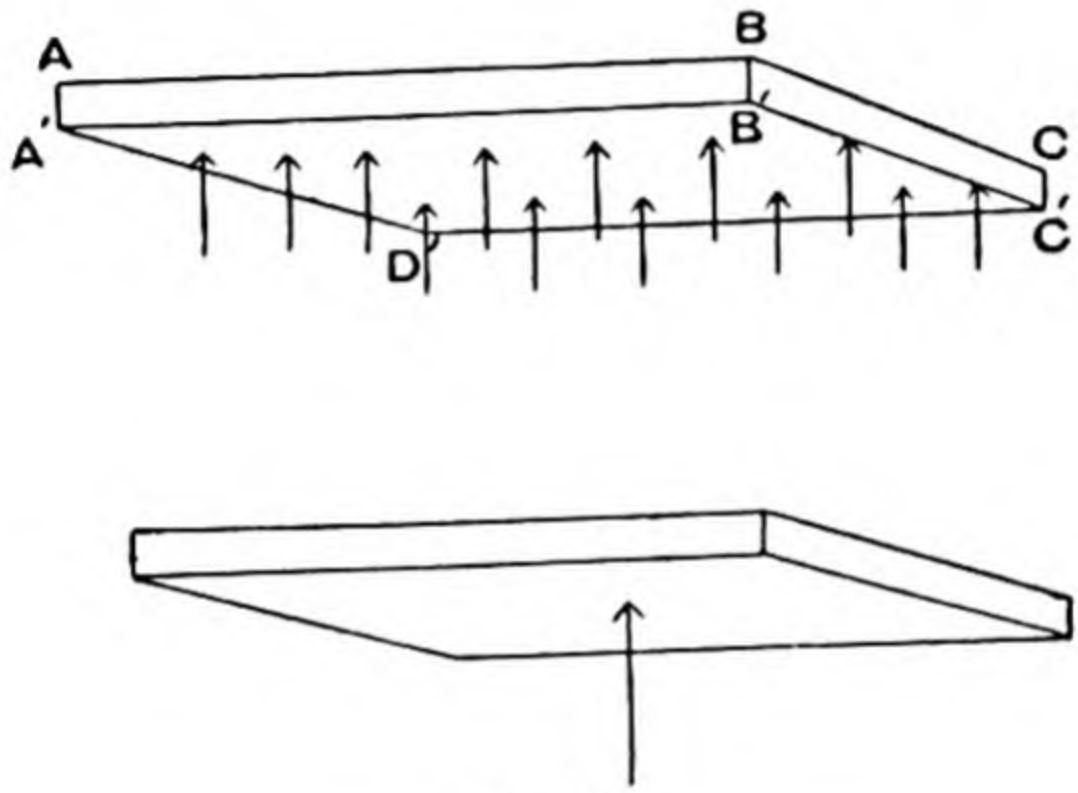


FIG. 11.

This pressure will act on each portion of the face of the block equally, and the total upward force will equal the weight of the block.

The block could, however, be kept in equilibrium if we had a single upward force acting at the mid point of the lower face, equal in magnitude to the weight of the block. This single force can therefore represent the combined effect of the several upward pressures, and it is called the *Resultant* Pressure.

If a gradually increasing upward force is applied at the

mid point of AD, the pressure of the table upwards on the lower face of the block will gradually diminish on the part

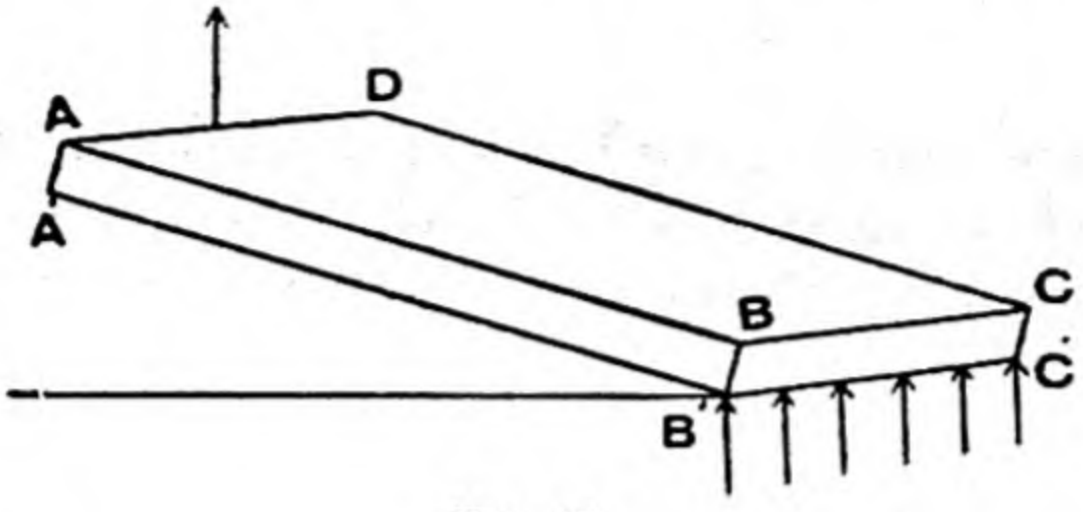


FIG. 12.

of the face towards A′D′, until, when that part leaves the table, the upward pressure will be entirely along the edge B′C′, and it could be represented by a single force at the mid point of B′C′.

# CHAPTER I.

## MOMENT OF A FORCE.

THE use of various mechanical instruments, such as the Lever, the Pulley, and the Inclined Plane, was known to the Egyptians, but we have no record of any scientific investigations as to the principle of these machines before the writings of Archimedes (287-212 B.C.).

Archimedes was born in Syracuse and studied at Alexandria. In his treatise he discusses the principle of the lever in the form of propositions similar to the geometrical proofs of Euclid.

He assumes as self-evident the statement that equal weights, acting at equal distances from their point of support, are in equilibrium; and that if the weights are equal, but act at unequal distances from the support, the one acting at the greater distance will sink.

His first assumption is the principle of the Balance. We find the weight of a body by balancing it against a known weight at the ends of a beam, pivoted so that the two arms are equal. When the beam remains horizontal we say that the weight of the body in one pan equals the known weight placed in the other pan.

Archimedes proceeds to prove theoretically from these assumptions, the relation which must exist between unequal

weights and their distances from the fulcrum, if they are to balance. His proof, although extremely ingenious, is very long and cumbrous, but no improvement was made upon it for 2000 years, when Stevinus of Bruges (1548-1620) gave a shorter proof on similar lines.

We shall find this relation by the following experiment.

*Experiment* 1. Balance a metre measure on a knife edge (a file), adjust two equai weights (10 grams), one on either side of the knife edge, until they balance.

FIG. 13.

(If the weight is circular, and one edge is at a distance $x$ from the end of the scale, the other edge being at a distance $y$ from the same end, prove that the centre of the weight is at a distance $\frac{x+y}{2}$.)

Double one of the weights and adjust the other, until the beam again balances. Measure the distance of the centre of each weight from the knife edge (called the fulcrum), and state the ratio between them.

We know that, if the weights were placed at the fulcrum, they would have no turning effect on the beam, and the experience of closing a door has taught us that the turning effect of the force we exert, depends on the distance from the hinge that the force is exerted.

If we consider that the turning power of a weight, at a given distance from the fulcrum, is doubled if the weight is doubled, we see from our experiment that the turning power of the weight can also be doubled by doubling its distance from the fulcrum.

*Experiment* 2. Balance a metre measure at its centre of gravity. Place 20 grams, with its centre at a distance of 10 cms. from the fulcrum, and find at what distance from the fulcrum, on the other side, a weight of 5 grams must be placed to balance it.

Then try 10 grams, 50 grams successively, in place of the 5 grams.

In each case multiply the weight by the distance of its centre from the fulcrum, and compare the results.

This product is called the *Moment* of the weight about the fulcrum, and we see that two weights will balance, provided their moments about the fulcrum are equal.

Many examples of this kind of lever will be familiar: a see-saw, a crowbar, each arm of a pair of pincers.

**Equilibrant and Resultant.**

*Experiment* 3. Tie a string through a hole at O, the middle of the top edge of a metre measure, and fasten it to a dynamometer. Read the dynamometer.

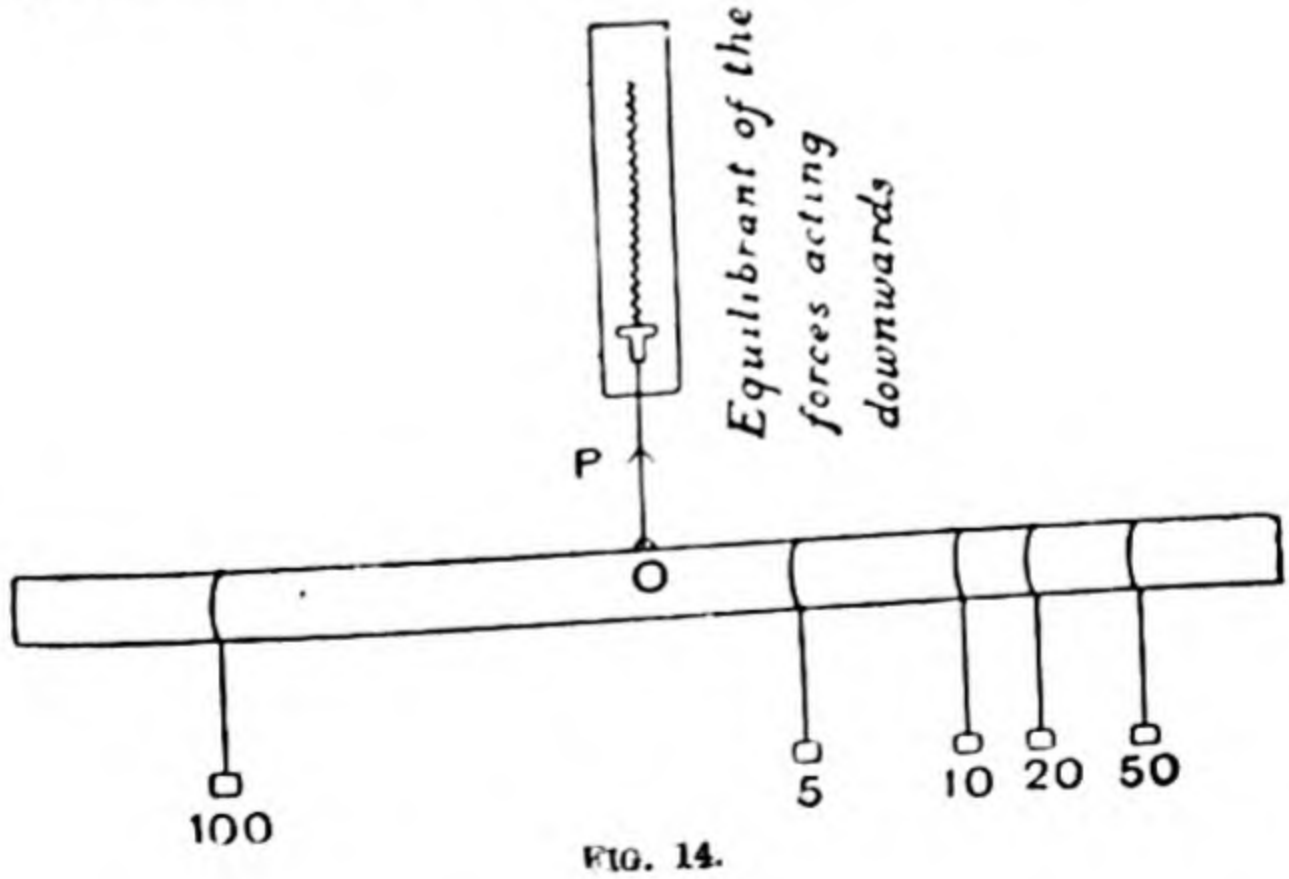


FIG. 14.

Hang 100 grams about 30 cms. from O, and on the other side of O adjust 5, 10, 20, 50 gram weights at various points, until the measure balances.

Again read the dynamometer, and compare the reading (P) with the forces acting downwards on the measure.

The single force P, exerted by the dynamometer, which neutralises the combined pull of the other forces, is called the *Equilibrant* of these forces.

If we substitute a single force for all the forces acting downwards, so that the downward pull on the dynamometer is unaltered, this single force is called the *Resultant* of the forces acting downwards.

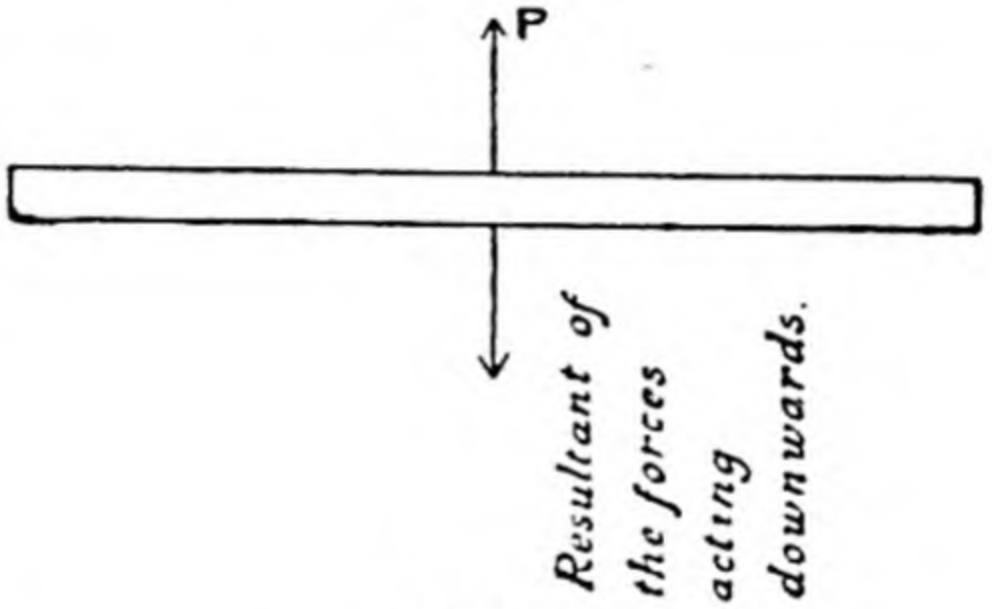


FIG. 15.

It is clear that this Resultant would be equal in magnitude to the Equilibrant, and would act in the same straight line, but in the opposite direction.

(If the metre measure were balanced on a file, instead of being supported by a dynamometer, the equilibrant P would be produced by the upward pressure of the file on the measure.)

Calculate the moment of each of the weights about O, and compare the sum of the moments of the weights on

the right-hand side with the moment of the 100 grams wt. on the other side.

EXAMPLE. A beam AB, 8 ft. long, will balance at a point C, $3\frac{1}{2}$ ft. from B. If the beam is supported at this point, and a weight of 10 lbs. is placed at B, find what weight must be placed at A to balance it.

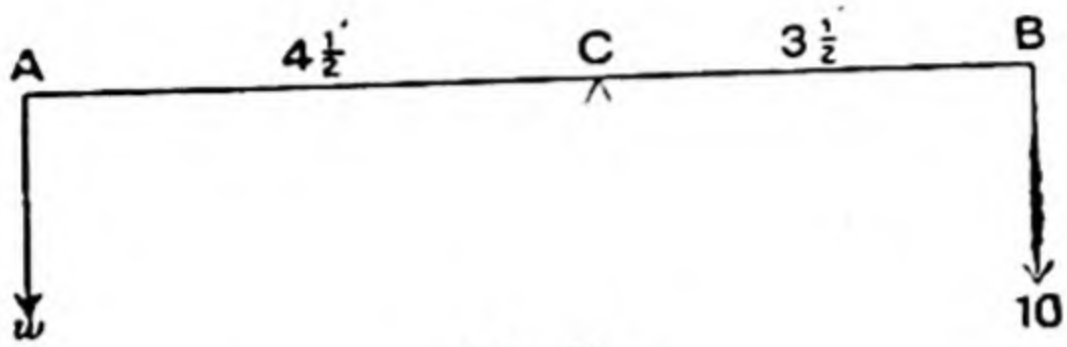


FIG. 16.

Since the beam balances at a point C, $3\frac{1}{2}$ ft. from the end B, the weight of the beam acts at the point C, which is the centre of gravity of the beam.

Let the required weight acting at A be $w$ lbs.

Since the beam is to balance about C, the moment of $w$ about C must equal the moment of 10 lbs. about C.

$$\therefore \quad \tfrac{9}{2} w = \tfrac{7}{2} 10.$$

$$\therefore \quad w = \tfrac{70}{9} = 7\tfrac{7}{9} \text{ lbs.}$$

## EXAMPLES I.

**1.** A uniform beam is balanced at its mid point. If a weight of 5 lbs. is placed on it, 2 ft. from this point, where must a weight of 8 lbs. be placed to maintain equilibrium ?

**2.** A uniform plank, 10 ft. long, is used as a see-saw, and can turn round an axis at its mid point. If a boy weighing 9 stone sits one foot from one end, where must a boy of 10 stone sit to keep the plank horizontal ? What is the pressure on the support, if the plank weighs 50 lbs. ?

**3.** A stick AB, 4 ft. long, will balance at a point C, 20 in. from A. If the stick is supported at this point, and a weight

of 5 lbs. is placed at A, find what weight placed at B will balance it.

**4.** A body placed in one scale pan of a balance is balanced by some shot placed in the other. A weight of 20 grams substituted for the body is found to balance the shot. What is the weight of the body? Borda's method (1733-1799).

**5.** If the left arm of the balance in Qu. 4 is $\frac{100}{101}$ part of the right arm, find what is the true weight of a body which, when placed in the left pan, is balanced by 60 grams in the right pan.

**6.** A pump handle, 4 ft. long, is pivoted at its centre of gravity, which is 6 in. from the end to which the piston rod is attached. What upward pull does the rod exert, if the other end of the handle is pushed down with a force of 20 lbs. weight?

**7.** The beam of a balance is horizontal when no weights are in the scale pans, but it is found that 20 grams in the left pan requires 20·1 grams in the other pan to balance it. How would you account for this? What weight in the left pan would balance 20 grams in the right pan?

**8.** A uniform bar, 4 ft. long, is pivoted at its centre. A weight of 10 lbs. is hung from one end, and the bar is kept horizontal by hanging another weight of 16 lbs. on it. Where should this weight be placed?

**9.** A rod AB is horizontal when supported at a point $2\frac{1}{2}$ ft. from the end A. A weight of 2 lbs. is now fastened to this end, and the rod remains horizontal if 3 lbs. is fastened to the other end. Find the length of the rod.

**10.** The arms of a balance are of unequal length, but it is pivoted at its centre of gravity. A body, when placed in the left pan, is balanced by $W_1$ grams in the right pan (this is the apparent weight of the body). If the body is placed in the right pan it is balanced by $W_2$ grams in the left pan. Find the true weight of the body. Gauss' method (1777-1855).

*Experiment* 4. We have seen that, if we wish to keep a metre measure in equilibrium in a horizontal position, we must place the fulcrum at G, the mid point of the scale, so that the upward thrust produced by the knife edge may neutralise the weight of the measure.

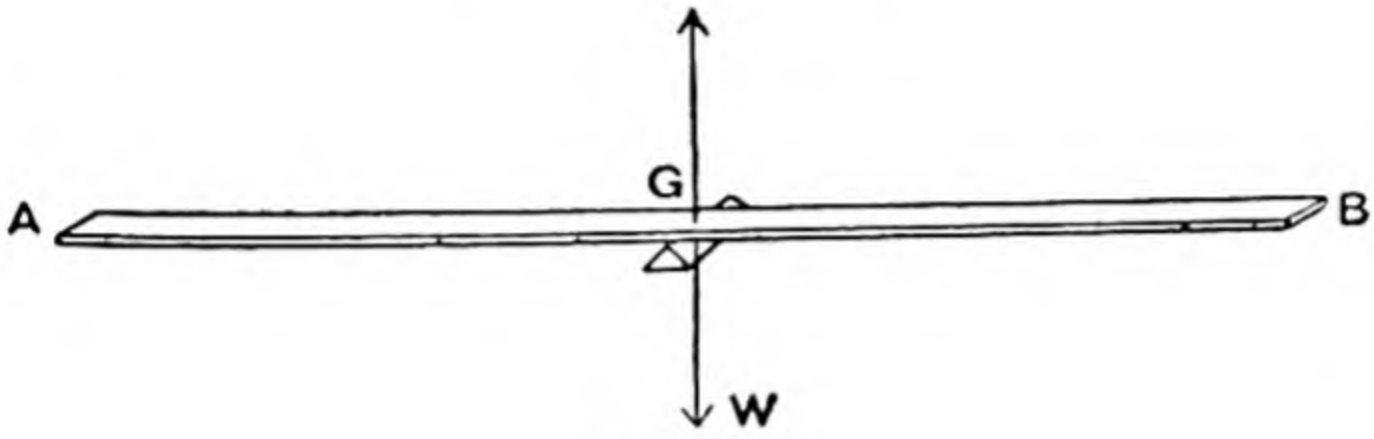


FIG. 17.

If we move the fulcrum to a point C, the weight of the measure, which is equivalent to a single force acting at G,

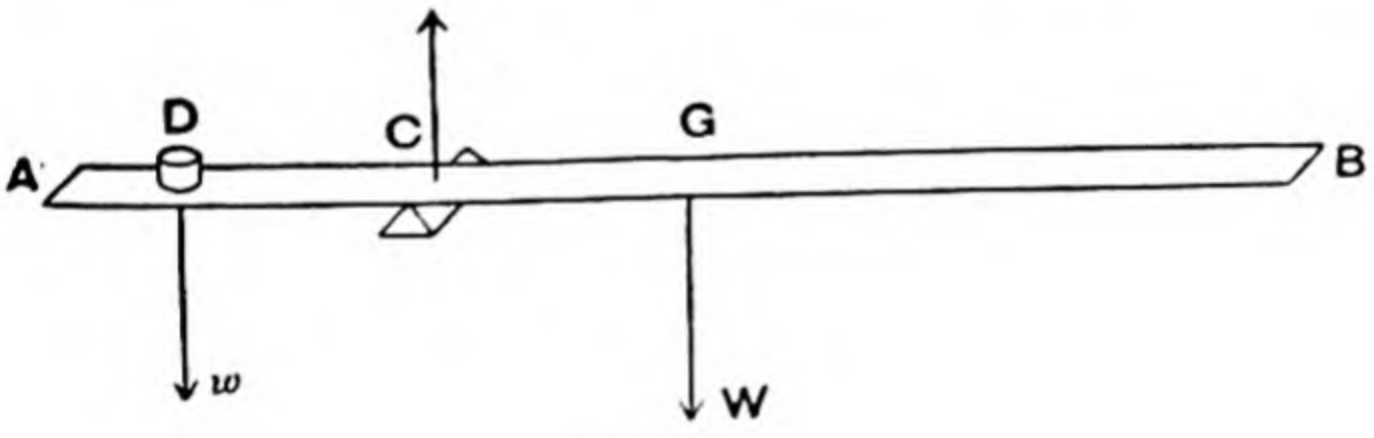


FIG. 18.

will produce a rotation of the measure round C. We can keep the measure in equilibrium by placing a weight $w$ at some point D between A and C.

Find, by experiment, the position of the point D, if $w = 80$ grams, and hence calculate the weight of the measure.

Since the weight of the portion CB of the scale tends to produce rotation in the opposite direction to that produced by the weight of AC, the beginner is apt to think it necessary to consider the moment of the weight of each part of the measure separately, instead of taking the moment of the weight of the whole measure acting at G.

Suppose the measure to weigh 100 grams, and suppose the fulcrum C to be 30 cms. from A.

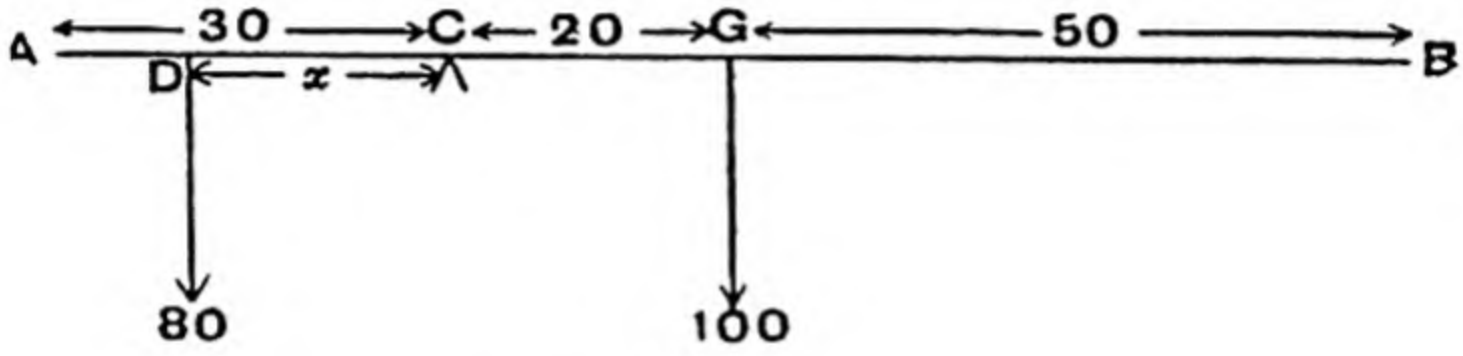


FIG. 19.

If CD $= x$ cms., we have

$$80\,x = 100(20)\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\therefore\ x = 25 \text{ cms.}$$

Now, consider the moment of the weight of each part of the measure separately.

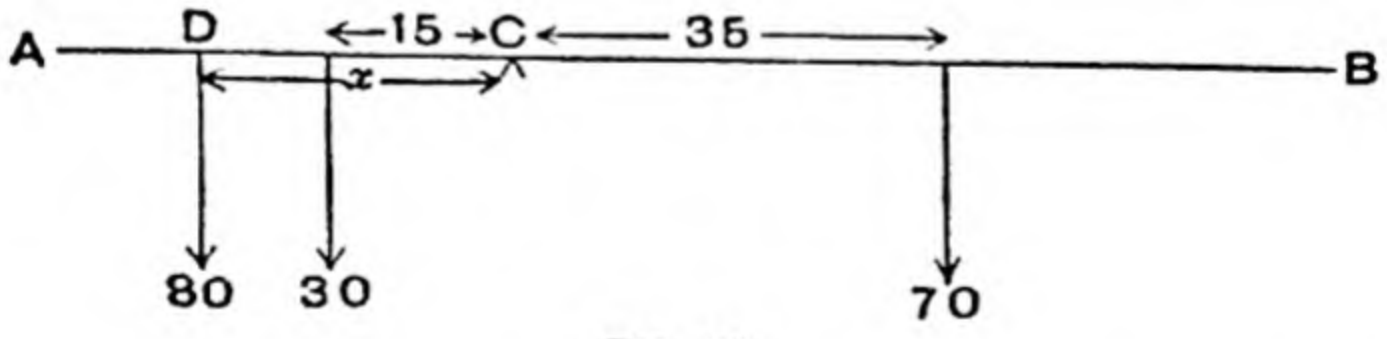


FIG. 20.

AC will weigh 30 grams, and CB will weigh 70 grams.

Then $$80x + 30(15) = 70(35),$$

*i.e.* $$80x = 70(35) - 30(15) = 2000,$$

which is the same equation as (i).

*Experiment* 5. Place one end of a metre measure on a knife edge at A, and support the measure in a horizontal position by a dynamometer fastened to the scale at B.

The force P tends to turn the measure round A in a counter-clockwise direction, and the weight of the measure tends to turn it round A in the opposite direction.

Calculate the moment of the weight of the measure about A, and compare the result with the moment of the force P about A.

Here the fulcrum is at one end of the lever, as is the case in the wheelbarrow, and the oar of a boat if we consider the blade to be fixed (approximately) in the water during a stroke.

Now, hang various weights at different parts of the scale. Calculate their moments about A (include the moment of

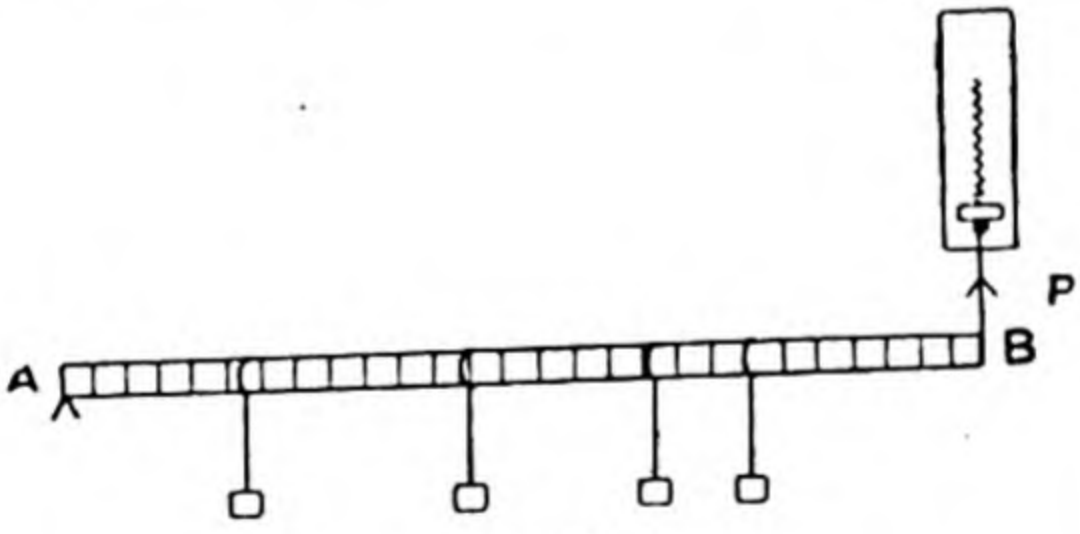


FIG. 21.

the weight of the scale), and compare their sum with the moment of the force P about A.

To find the force exerted by the fulcrum at A, substitute a dynamometer for the knife edge at A, and note that, if the reading is Q, then P + Q equals the sum of the forces acting downwards on the scale.

It is evident that this experiment is a simple case of a very common type of problem such as the following:

A span of a bridge is supported on piers at its two extremities; find the pressure on the piers when the bridge has loads placed at various points on it.

The upward pressures of the piers on the bridge are represented by the forces P and Q, and can be found by taking moments as in the experiment. The pressure of the bridge on the piers is the same as the pressure of the piers on the bridge, but in the opposite direction.

When working examples in Statics, the following points must always be clearly stated:

1. The body (or portion of a body) whose equilibrium is being considered.
2. The forces acting on that body.
3. The principle which is being applied.

EXAMPLE 1. A rod AB, 100 cms. long, whose weight, 50 grams, acts at a point G, 60 cms. from A, is supported at C

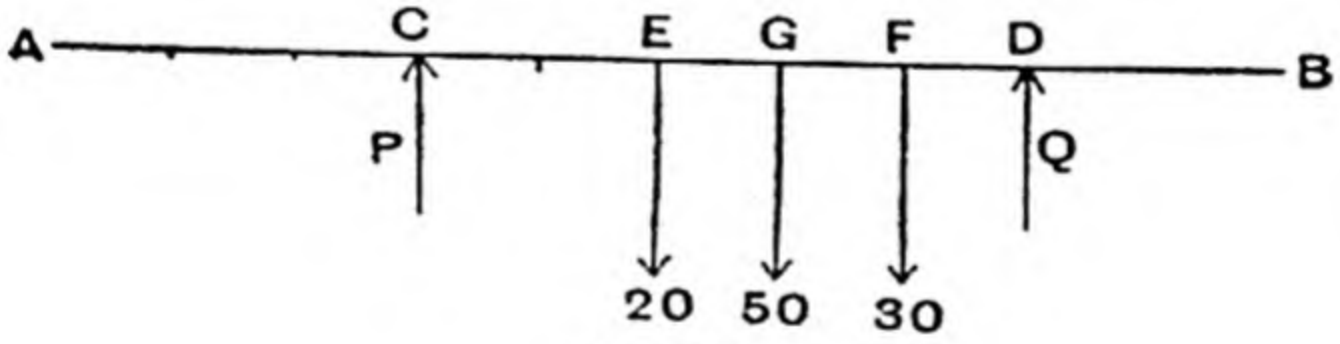


FIG. 22.

and D, and has weights of 20 grams and 30 grams at E and F respectively. If AC = 30 cms., AE = 50 cms., AF = 70 cms., AD = 80 cms., find the pressures on the supports.

We consider the forces acting on the rod.

The rod is in equilibrium under the action of its weight, the forces 20 grams wt. and 30 grams wt. vertically downwards, and the pressures P and Q *of the supports on the rod* vertically upwards.

Taking C as a fulcrum, we have, by the principle of moments,

$$20(20) + 50(30) + 30(40) = Q(50), \quad i.e.\ 3100 = 50Q\ ;$$

$$\therefore\ Q = 62 \text{ grams wt.}$$

Since $P + Q = 20 + 50 + 30 = 100$, $\therefore\ P = 38$ grams wt.

Now, the pressure P of the support on the rod at C is equal to the pressure of the rod on the support at that point. Similarly at D.

Therefore the pressures on the supports are 62 grams wt. at D and 38 grams wt. at C.

EXAMPLE 2. A uniform bar AB weighs 10 lbs. and is 6 ft. long. If a weight of 2 lbs. is placed at A and a weight of 4 lbs. at D, 1 ft. from B, find at what point the bar will balance.

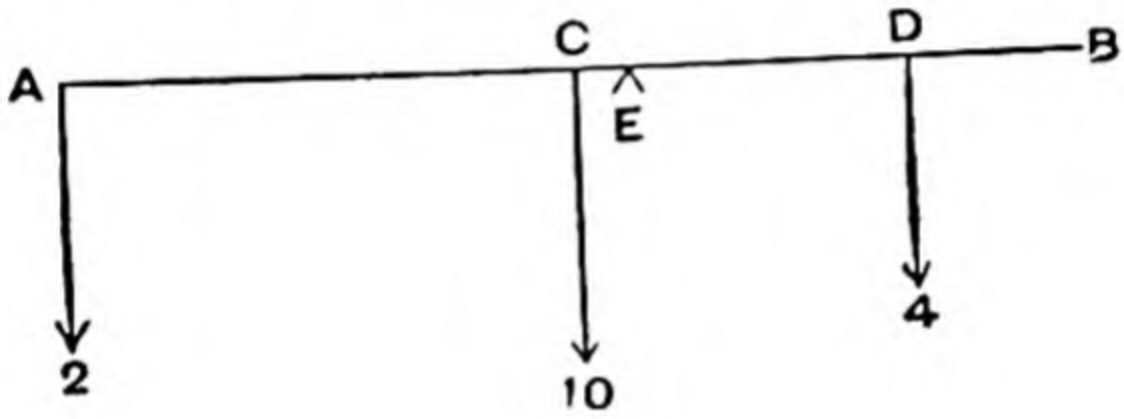


FIG. 23.

Let the bar balance at E, where $AE = x$ ft.

The bar is in equilibrium under the action of its weight, the two forces 2 lbs. and 4 lbs. vertically downwards, and the upward pressure of the fulcrum at E.

Taking moments about E, we have

$$2x + 10(x - 3) = 4(5 - x)\ ;$$

$$\therefore\ 16x = 50\ ;$$

$$\therefore\ x = 3\tfrac{1}{8} \text{ ft.}$$

The bar will balance at a point $3\frac{1}{8}$ ft. from A.

## EXAMPLES II.

**1.** A uniform bar, AB, 6 ft. long, weighing 60 lbs., is fastened at C, a point 2 ft. from A, in a horizontal position, so that it cannot move. What is the moment about C of (1) the weight of the part AC of the bar; (2) the weight of the part CB; (3) the weight of the whole bar? What relation is there between these three results?

**2.** A Jack in the box pushes with a force of $\frac{1}{2}$ lb. weight at the mid point of the lid, which is 12 cms. long. What must be the least force exerted by the catch to keep the box closed?

**3.** A uniform plank AB, 10 ft. long, weighs 100 lbs. It rests on two vertical supports, one at A and the other 3 ft. from B. If a boy weighing 9 stone, standing on the plank, makes it tilt, find where he is standing.

**4.** A uniform bar, 100 cms. long, balances at a point 30 cms. from one end when a 100 gram weight is placed 15 cms. from that end. Find the weight of the bar.

**5.** A light bar, 5 ft. long, whose weight may be neglected, is supported by two vertical strings at its ends. If the string breaks when its tension is more than 6 lbs., find how near to the end of the bar a weight of 10 lbs. may be placed.

**6.** A uniform bar AB, 5 ft. long, weighing 10 lbs., is supported by two vertical strings, one at A and the other 1 ft. from B. Find what weight can be placed at B before the string at A becomes slack.

**7.** A uniform plank, 12 ft. long, weighing 100 lbs., rests across two parallel horizontal poles placed the same height from the ground at right angles to the plank. The end of the plank projects one foot beyond one pole, and the other end projects 2 ft. beyond the other pole. What is the weight of the heaviest man who can walk along the plank without tilting it?

**8.** A uniform rod, 6 ft. long, rests on a table, and its end is pushed over the edge. How much of the rod can project over the edge before the rod will topple?

**9.** A uniform beam AB, 7 ft. long, rests horizontally on two supports 1 ft. and 5 ft. from A. If the greatest weight

that can be hung from B without tilting the beam is 15 lbs., find the weight of the beam. Find also what weight can be hung from A without upsetting the beam when there is no weight at B. Find the pressure on the supports in each case.

**10.** A uniform crowbar, 4 ft. long, weighing 12 lbs., is used to lift a block of stone. If the edge of the block produces a vertical pressure of 1 cwt. on the crowbar, 1 ft. from the end, find what vertical force a man must exert at the other end to move the block.

**11.** A beam AB, 4 ft. long, weighs 10 lbs., and balances at a point $2\frac{1}{2}$ ft. from A. If a weight of 2 lbs. is placed at A and 3 lbs. at B, where will the beam now balance ?

**12.** A uniform bar AB has a weight of 5 lbs. suspended at B, and will remain horizontal if a fulcrum is placed 2 ft. from B. If 4 lbs. is added to the weight at B the fulcrum must be moved 6 in. towards B to maintain equilibrium. Find the length of the bar and its weight.

**13.** A ladder, 16 ft. long, weighing 80 lbs., has its centre of gravity 6 ft. from the bottom. Two men carry it in a horizontal position, one of them 3 ft. from the bottom, the other 6 ft. from the top. Which man exerts the greater pressure, and by how much ?

**14.** A lever safety valve ABC turns about a fulcrum at A. The resultant thrust of the steam on the valve acts vertically upwards at B, and the valve is held down by a weight applied at C. If AB = 3 in. and BC = 20 in., calculate the weight at C, if the valve may just blow when the upward thrust of the steam is 700 lbs. wt.

**15.** A bar of uniform section, 12 ft. long, weighing 20 lbs. per foot, rests on two supports 9 ft. apart (the ends projecting equally beyond the supports). It carries a load of 300 lbs. at a distance of 3 ft. 6 in. from the left-hand support. Calculate (1) the pressure on each support due to the load carried ; (2) the pressure on each support due to the combined effect of the load and the weight of the bar.

**16.** AB, a uniform bar 8 ft. long, weighing 150 lbs., is supported in a horizontal position on two vertical props at A and B. How near an end may a load of 200 lbs. be placed, if the pressure on a prop may not exceed 250 lbs. ?

**17.** A uniform rod, 6 ft. long, weighing 10 lbs., has weights of 1 lb., 5 lbs., 6 lbs., and 4 lbs. suspended at points 1 ft., $2\frac{1}{2}$ ft., 5 ft., and 6 ft. respectively from one end. Find at what point the rod will balance.

**18.** A uniform bar, 14 ft. long, weighing 160 lbs., is supported on two props, 2 ft. and 4 ft. from either end respectively. A string is attached to one end from a point vertically above it. What tension must be put on the string if the two props are to support equal weights? To which end must the string be attached?

**19.** A uniform plank AB, 20 ft. long, and weighing 180 lbs., rests horizontally on two supports, one at A and the other at C, 15 ft. from A. A rolling weight of 300 lbs. can be moved along the plank. Find expressions for the pressures on the two supports when the rolling weight is $x$ ft. from A. Plot a graph showing how the pressure on A varies as the rolling weight is moved from A towards B, and find the position of the weight when the plank just lifts off the support A.

**20.** A uniform plank AB, 5 ft. long, weighing 60 lbs., rests in a horizontal position on two supports, one at A and the other 2 ft. from B. Another similar plank CD is laid across AB at right angles at E, 1 ft. from A, and is supported by a man at D. If E is 2 ft. from C, find the pressure on the supports and on the man.

## The Steelyards.

*The Roman Steelyard.* This lever, which is still in common use, is supported at a point B, and the weight of a body

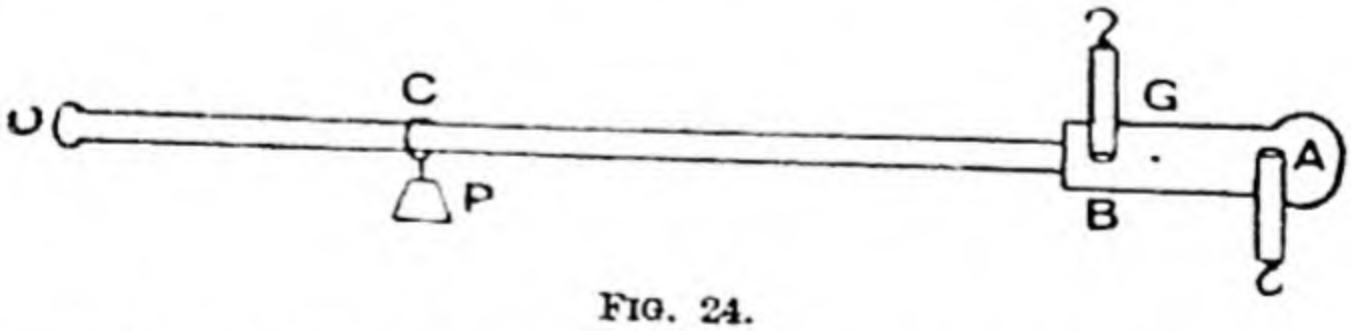


FIG. 24.

suspended at A is shown by the reading on the graduated bar DB at C, where the movable constant weight P has to be placed to produce equilibrium.

**21.** The weight of the lever is 4 lbs., and acts at G. The movable weight P is 6 lbs. BD = 2 ft., BG = 3 in., BA = 6 in. Find :

(i) The position of the graduation marked 0, *i.e.* the position of P when there is no weight at A.

(ii) The distance between the graduations on BD showing weights of 1 lb., 2 lbs., 3 lbs. at A.

(iii) The greatest weight that the steelyard can weigh.

*Danish Steelyard.* In this instrument the weight of the bar acts near one end at G, and the fulcrum B is adjusted until the weight at A is balanced. The graduation on the bar at B shows the weight at A.

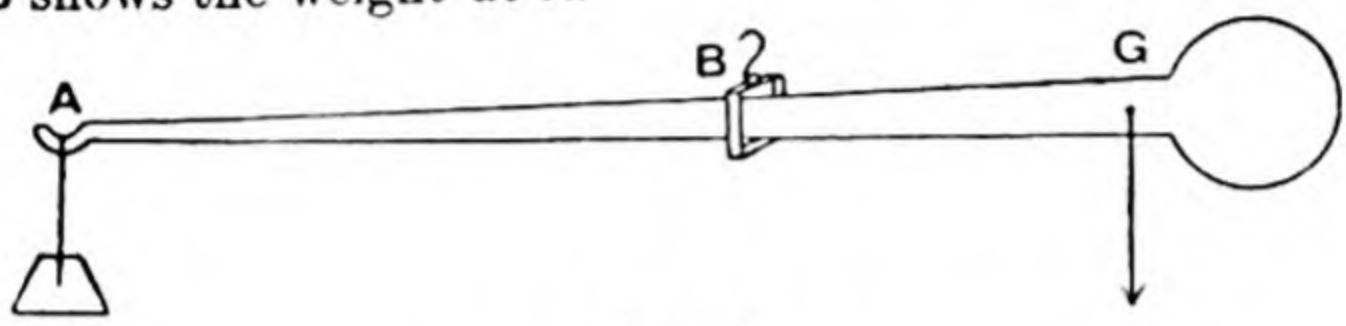


FIG. 25.

**22.** The weight of the lever is 6 lbs. and the distance AG = 4 ft. Find :

(i) The position of the zero graduation.

(ii) The distance from G of the graduations showing 1 lb., 2 lbs., 3 lbs. at A, *i.e.* the position of the fulcrum B when these weights are placed at A.

(iii) The weight at A when B is at a graduation $2\frac{1}{2}$ ft. from G.

**23.** The movable weight o. a Roman steelyard is lost, but it is found that the graduations reading 1 lb., 2 lbs., etc., are 1 in. apart. Show that the missing weight may be calculated without knowing the weight of the lever or its centre of gravity.

**24.** When 1 lb. is hung at the end of a Danish steelyard the fulcrum is 2 ft. from that end. If 2 lbs. is substituted for the 1 lb., the fulcrum has to be moved 1 in.; find the weight of the lever.

**25.** The bar of a common steelyard is 2 ft. long, weighs 3 lbs., and is suspended at a point 2 in. from one end, and its C. of G. is distant 3 in. from the same end. Find the greatest weight that can be weighed with this steelyard, the movable weight being 5 lbs.

**Moment of a Force which does not act at Right Angles to the Lever.**

*Experiment* 6. Cut out from a piece of stiff cardboard a circular disc of radius about 10 cms. Fasten it to a vertical board, so that it can turn easily about its centre.

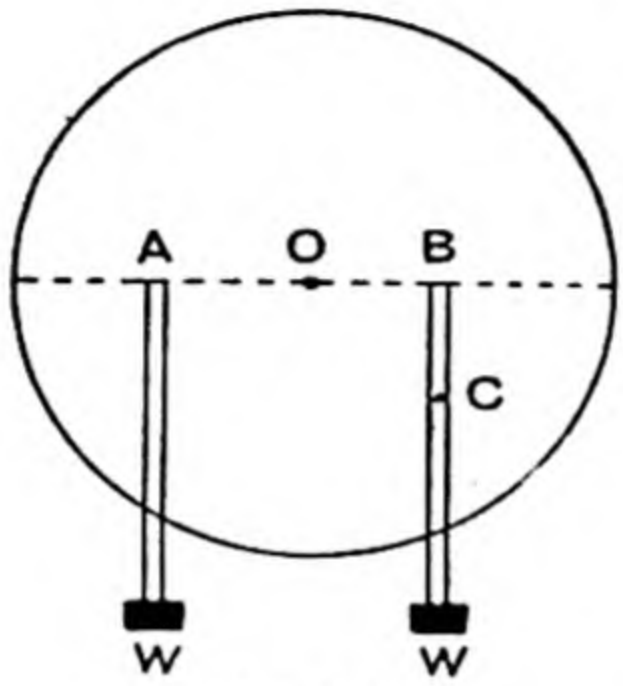


FIG. 26.

Attach two equal weights to two pieces of tape, fasten the tapes to the cardboard by drawing pins, and adjust them on a diameter, until the disc is in equilibrium.

If we move one pin from B to C, a point vertically below B, will the disc turn round ?

Find the locus of this pin, for the disc to remain in equilibrium.

Replace the pin at B. If the disc is turned through an angle of say 60°, will it remain in equilibrium ? Try it.

Now make the weights unequal, and adjust the tapes on the disc, so that there is no rotation.

If they balance when $W_1$ is at A and $W_2$ at B, it will be found that equilibrium is also maintained if $W_1$ is fastened at any point in the vertical line through A, and $W_2$ at any point in the vertical line through B.

If, then, $W_2$ is fastened at $B_1$, its turning power is the same as when fastened at B, so that its moment is still $W_2 \times OB$, and not $W_2 \times OB_1$.

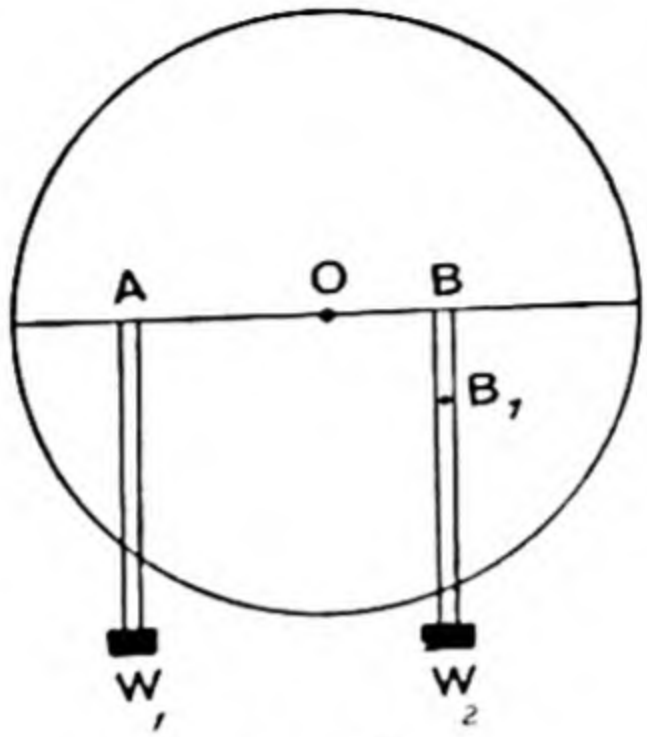


FIG. 27.

It will be clear from this experiment that the condition for the weights to balance is that their moments about an axis at O shall be equal; the moment of a force about a point O being now properly defined as the product:

*Force* × *perpendicular drawn from the point* O *to the line of action of the force.*

**Further Illustration of the same Principle.**

*Experiment* 7. Support a half-metre measure by a needle through a hole at the mid point C.

Hang a 50 grams weight at one end, and fasten a string to B and to a dynamometer, so that the direction of the pull P makes an acute angle with AB. A stand D will be required to prevent the scale from swinging round.

The moment of the 50 grams weight about the axis at C is $50 \times AC$. If we read the dynamometer, we find that

the reading P (corrected for the position of the dynamometer) is not 50 grams.

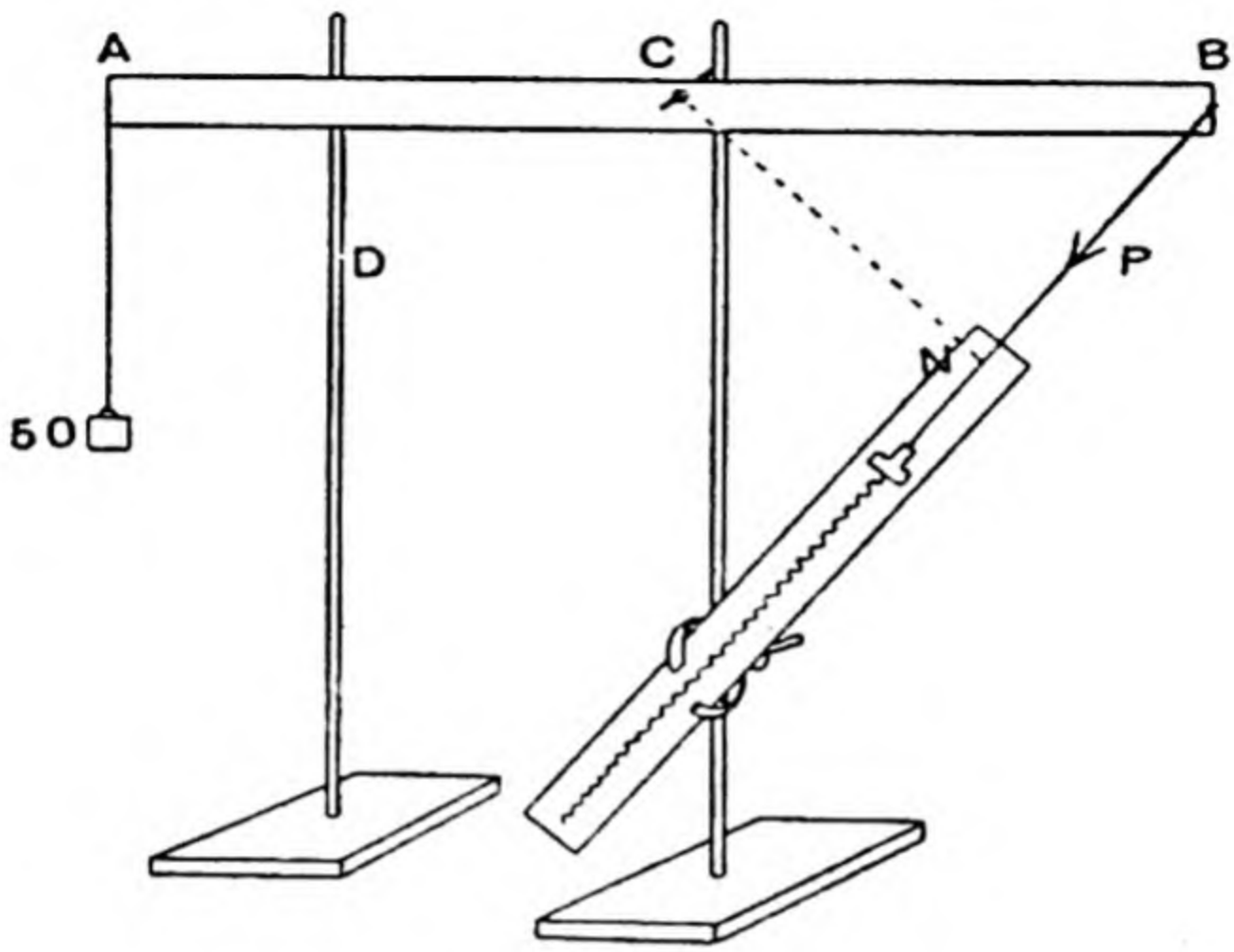


FIG. 28.

Therefore P × BC does not measure its moment.

Drop a perpendicular CN from C to the line of action of P, and we find that P × CN is approximately equal to 50 × AC, so that the turning power or moment of P is measured by P × CN.

Instead of a measure, a piece of stiff cardboard of any shape may be pivoted at its C. of G., so that it may move easily in a vertical plane. Two strings are then fastened to it and pulled (in a vertical plane) by dynamometers. It will be found that when there is equilibrium $T_1 p_1 = T_2 p_2$, where $p_1$, $p_2$ are the perpendiculars from the fulcrum to the line of action of the strings whose tensions are $T_1$, $T_2$ respectively.

**Historical Note.**

Leonardo da Vinci (1452-1519), the Italian painter, seems to have been the first to investigate the condition for equilibrium when the forces do not act at right angles to the arm of the lever. He says:

"We have a bar AD free to rotate about A, and, suspended from the bar a weight P, and, suspended from a string which passes over

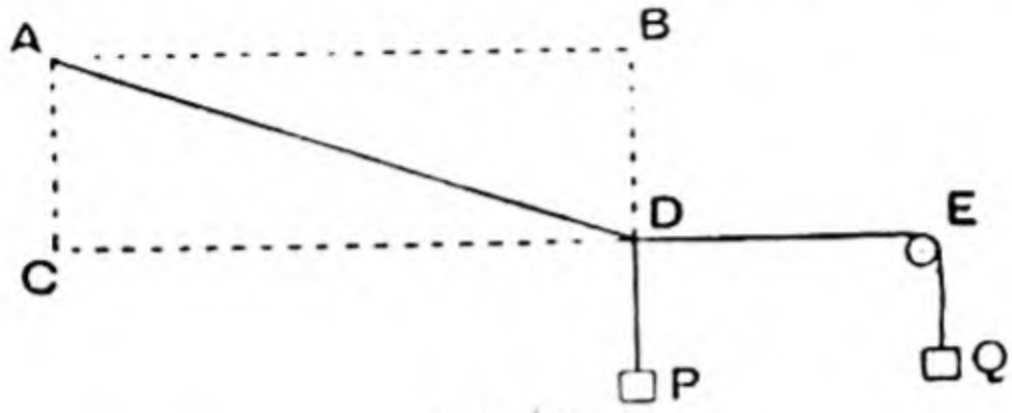


FIG. 29.

a pulley, a second weight Q. The lever arm for the weight P is not AD, but the 'potential' lever AB. The lever arm for the weight Q is not AD, but the 'potential' lever AC."

The direction of the pull on the rod caused by the weight Q is in the direction DE.

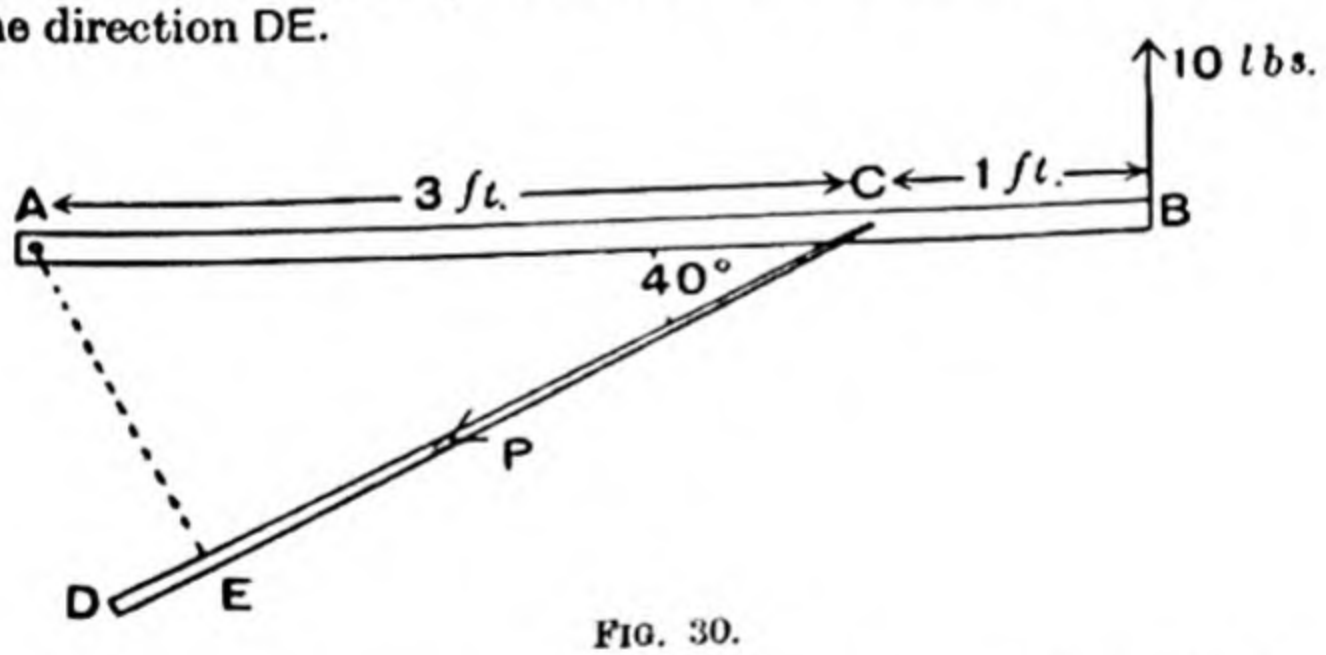


FIG. 30.

EXAMPLE. A bar AB, 4 ft. long, can turn in a horizontal plane round an axis at A. It is pulled at B by a horizontal force of 10 lbs. at right angles to the bar, which is kept from moving by a fixed horizontal rod fastened at the bar at C, 3 ft. from A, and inclined to it at 40°. Find the pull in this rod.

The bar is acted upon by a force of 10 lbs. at B, the pull P of the rod CD, and a force at A.

Since there is no rotation in the horizontal plane round A, the moment of the 10 lbs. about A must equal the moment of P about A.

Drop a perpendicular AE from A to CD, then $AE = 3 \sin 40$. Therefore, by the principle of moments,

$$10 \times 4 = P \times 3 \sin 40 ;$$

$$\therefore\ P = \tfrac{4}{3}\cdot 0 \operatorname{cosec} 40 = 20{\cdot}7 \text{ lbs.}$$

## EXAMPLES III.

**1.** To turn a gate on its hinges, the least possible force required is 10 lbs. How and where should it be applied? If the line of action of the force were turned through 45° in a horizontal plane, what force would then be necessary?

**2.** A uniform beam is pivoted at its centre. If 10 lbs. hang at one end, what horizontal force at the other end will keep the beam inclined at 20° to the horizontal?

**3.** A uniform bar, 6 ft. long, weighing 8 lbs., can turn in a vertical plane round one end. What force at the other end acting at right angles to the bar will keep the bar inclined at 30° to the vertical?

**4.** A circular disc, 1 ft. radius, can turn easily in a vertical plane round its centre O. A weight of 2 lbs. is fastened on the rim at A. What horizontal force in the plane of the disc, acting at B, will keep OA horizontal, if OB is at right angles to OA and 4 ins. long?

**5.** What horizontal force would be required at B, if the disc in Qu. 4 were turned so that OA is inclined at 30° to the vertical?

**6.** A uniform bar AB, 4 ft. long, is hinged to a wall at A, and is kept horizontal, at right angles to the wall, by a string fastened to B, and to a point in the wall 5 ft. vertically above A Find the tension in the string if the bar weighs 300 grams.

**7.** A weight of 10 lbs. is hung from a nail by a string 5 ft. long. It is then pulled by a horizontal force until it is 3 ft. from the vertical line. What must be the magnitude of the

force? What force would be required if it acted at right angles to the string in the new position?

**8.** A uniform rod, 100 cms. long, weighing 80 grams, can turn in a vertical plane round a pin at one end. It is held at an angle of 22° to the horizontal by a force at its lower end, which is inclined at 50° to the vertical. Find this force (1) by measurement from a figure drawn to scale; (2) by calculation.

**9.** A small window, 2 ft. long, turns round a hinge at its lower edge, and is kept open at an angle of 40° to the vertical by a cord fastened to the mid point of its upper edge, which passes over a small pulley in the top framework. What will be the approximate tension in this cord if the window weighs 10 lbs.?

**10.** A weight of 3 lbs. is attached to a rim of a wheel at A. A weight of 1 lb. hangs from a string coiled round the circumference of the wheel. If the wheel can rotate in a vertical plane round an axis at its centre O, find the position of equilibrium, *i.e.* find the angle OA makes with the horizontal.

**11.** A rectangular block of stone, weighing 2 tons, stands on its end, which is 2 ft. square. If it is 5 ft. high, find what horizontal force at right angles to its top edge will tilt it. Consider the forces acting in a vertical section through the centre of gravity.

**12.** Find graphically what force acting at the same point horizontally would be required to support the block in Qu. 11 when it had turned through 20°.

**13.** A trap door ABCD is 4 ft. long and weighs 12 lbs. It opens round a hinge along AB, and is propped up by a bar in a vertical plane 2 ft. long, one end of which is under the mid point of CD and the other on the edge of the horizontal frame. Find the thrust in this bar.

**14.** A uniform rectangular board ABCD, 4 ft. by 1 ft., weighing 2 lbs., is pivoted at the mid point of the longer edge AB, so that it can turn in a vertical plane. Find what weight hung at B will make the board turn through 20°.

**15.** If a weight of $\frac{1}{4}$ lb. had been hung at B (see Qu. 14), find through what angle the board would have turned.

**16.** The beam of a balance turns about B, the mid point of its top edge AC, where $AB = BC = l$. The C. of G. of the beam, whose weight is W, is at G, vertically below B, when AC is horizontal, where $BG = l'$. If a weight P is suspended at A and a weight $P + p$ at C, and the beam turns through an angle $\alpha$, prove $\tan \alpha = \frac{pl}{Wl'}$.

**17.** A pole is fastened vertically in the ground, so that 10 ft. is above the surface. A rope fastened to the top and to the ground, 20 ft. from the pole, will break the pole at the ground if stretched with a force of 1 cwt. What force would break the pole if the rope is tautened at double the distance from the foot ?

**18.** The axle of a capstan is 2 ft. in diameter. If four men push, each with a force of 40 lbs. weight, at right angles to the bars of the capstan, at a distance of 6 ft. from the axis, find the greatest weight lifted by a rope coiled round the axle of the capstan.

**19.** A uniform solid hemisphere, with a particle attached to a point of its rim, rests in equilibrium on a horizontal plane, with its plane face inclined to the horizon at an angle of 60°. Assuming that the distance of the C. of G. of the hemisphere from its centre is equal to $\frac{3}{8}$ of the radius, find the ratio of the weights of the hemisphere and the particle.

**20.** A rectangular plate of metal ABCD weighs 500 grams. It can turn freely in a vertical plane round A. Find what weight must be suspended from B to make the diagonal AC turn through 20°. $AB = 2$ ft.; $BC = 1$ ft.

**Parallel Forces.**

We have seen that the equilibrant of two parallel forces P and Q, acting on a body at A and B respectively, is a force P + Q acting at O, where

$$P \cdot AO = Q \cdot BO.$$

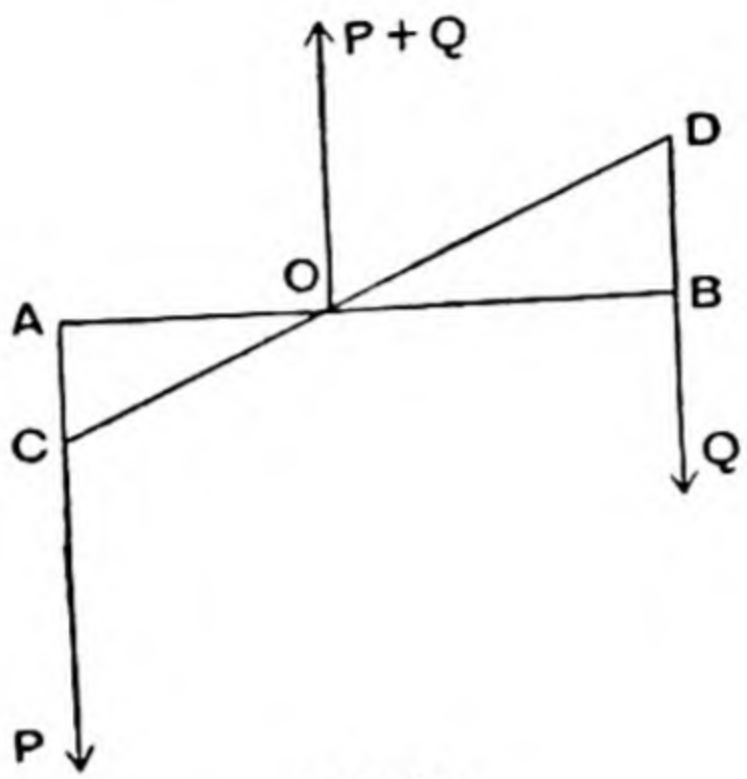


FIG. 31.

If, through O, we draw any line COD, we have seen that the forces P and Q, acting at C and D, will be in equilibrium, since the condition $P \cdot AO = Q \cdot BO$ is still satisfied.

If the angle $AOC = \theta$, we have therefore

$$P \cdot AO \sec\theta = Q \cdot BO \sec\theta,$$

or

$$P \cdot CO = Q \cdot DO.$$

Hence a body acted upon by the two parallel forces P and Q, at the points C and D respectively, will be kept in equilibrium by a parallel force P + Q, whose line of action cuts CD at O, where $P \cdot CO = Q \cdot DO$.

Since P, Q and P + Q are in equilibrium, it follows that Q at D is the equilibrant of P at C and P + Q at O.

Now, since $P \cdot CO = Q \cdot DO$, we have

$$P \cdot CO + P \cdot OD = Q \cdot DO + P \cdot OD,$$

*i.e.* $$P \cdot CD = (P + Q) \cdot OD.$$

Let P be called X, and let $P + Q$ be called Y. Then the equilibrant of X at C and Y at O is a force $Y - X$ which cuts CO produced on the side of the greater force at a point D, such that $X \cdot CD = Y \cdot OD$.

When parallel forces act in the same direction, they are said to be "Like parallel forces"; when they act in opposite directions, they are termed "Unlike parallel forces."

**Couples.**

A pair of *equal* parallel forces, acting in opposite directions, is called a *Couple*. Suppose a rigid body to be acted upon by a pair of equal forces P, in opposite directions. Take any point O in the plane of the forces.

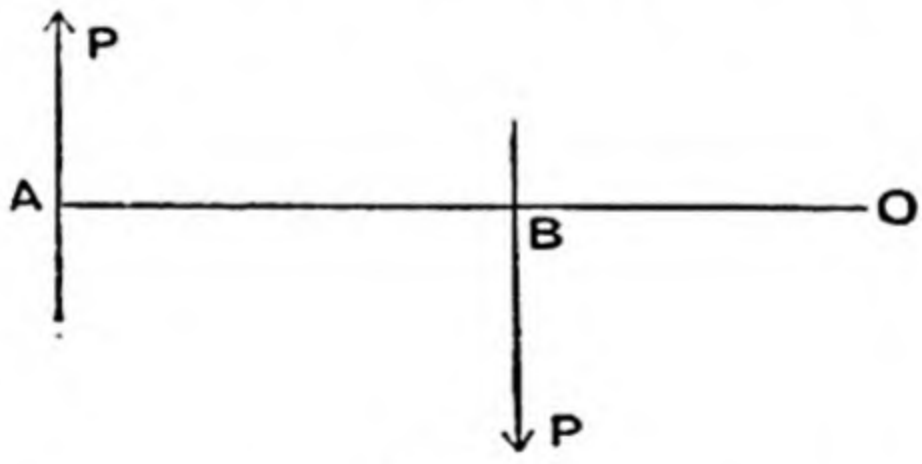


FIG. 32.

Draw OBA perpendicular to their lines of action.

Since the forces tend to turn the body round O in opposite senses, the total turning effect is measured by

$$P(AO) - P(BO) = P(AO - BO) = P \cdot AB.$$

*This result is independent of the position of* O.

The turning effect of a couple is called its *torque*, and is

measured by P . AB, the moment of the couple. AB is called the *arm* of the couple.

Take the point O between the lines of action of the forces, and show that the moment of the couple is still measured by P . AB.

No single force acting on a body can neutralise the effect of a couple. To keep the body in equilibrium, we must apply to it another couple of equal moment acting in the opposite sense.

The two forces applied by the fingers to turn a tap form a couple. The propeller shaft of a steamer is turned by a torque produced by the engines.

EXAMPLE. The ends of a compass needle, free to move in a horizontal plane about a vertical axis at its centre, are acted upon by the earth's magnetism, which produces constant

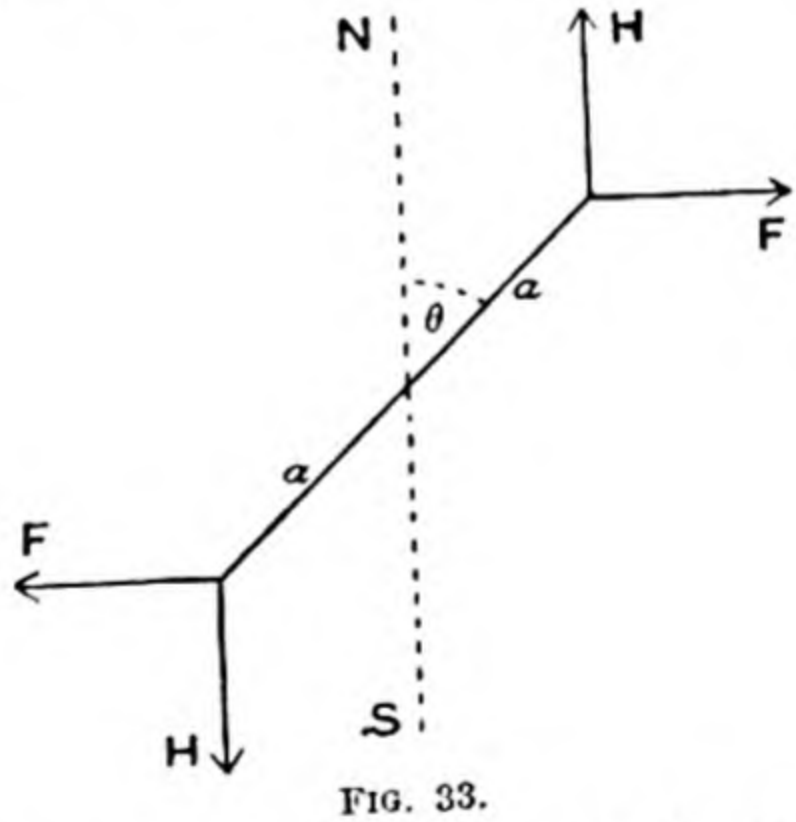


FIG. 33.

equal forces, due N. on one end and due S. on the other. A magnet in the plane of the needle, placed E. and W. in a certain position, will produce equal and opposite forces on the ends of the needle, as in Fig. 33.

Let H be the force produced by the earth, and F the force produced by the magnet on each end of the needle.

In the position of equilibrium, let the needle make an angle $\theta$ with the N. and S. line. The torque produced by the earth is $H \cdot 2a \sin\theta$, where $2a$ is the length of the needle. The torque produced by the magnets is $F \cdot 2a \cos\theta$. Since these are equal,

$$F \cdot 2a \cos\theta = H \cdot 2a \sin\theta ;$$

$$\therefore F = H \tan\theta.$$

## EXAMPLES IV.

**1.** A force of 8 lbs. wt. acts along a diagonal of a square lamina, and a parallel force of 3 lbs. wt. acts at an adjoining corner. Find the magnitude of their equilibrant and the point in which its line of action cuts the other diagonal (i) when the forces are like, (ii) when the forces are unlike.

**2.** Find the magnitude and the position of the equilibrant of two unlike parallel forces P and Q, whose distance apart is 10 cms., (i) when $P = 100$, $Q = 90$; (ii) when $P = 100$, $Q = 99$; (iii) when $P = 100$, $Q = 100$.

**3.** Two parallel forces P and Q act at A and B respectively. Prove that if both forces are turned through the same angle, the point in which AB is cut by their equilibrant will remain unaltered.

**4.** A uniform beam weighing 10 lbs. per foot-length projects 6 ft. at right angles to a vertical wall into which it is fastened. Where does the vertical upward force act on this part of the beam, in order to neutralise its weight? Show that the beam is acted upon by a couple which tends to break it and find its moment.

**5.** P and Q are two unlike parallel forces. P is 12 lbs. wt., and acts at A; Q is 8 lbs. wt., and acts at B, 2 ft. from A, each force making an angle of 60° with AB. Draw AC, 8 units in the direction of P, and BD 12 units in the same direction as AC. Join DC, and produce to meet BA produced in E. Prove that E is a point on the line of action of the resultant of P and Q.

**6.** Give the corresponding construction when P and Q are like parallel forces.

**7.** Along the sides AB and CD of a square ABCD are forces each equal to 3 lbs. weight. If E is the mid point of AB and F the mid point of CD, what forces acting along EF and CB will maintain equilibrium ?

**8.** A compass needle, free to turn in a horizontal plane, is acted upon by the earth's magnetism, as in Fig. 33. If the equal and opposite forces F act at right angles to the needle and the needle makes an angle $\theta$ with the N. and S. line when in equilibrium, prove that $F = H \sin \theta$.

## Additional Laboratory Experiments.

**1.** Support a beam near one end by a knife edge resting on the table, and near the other end by a spring balance hooked into an eye on the beam. Keep the balance vertical by clamping it in a retort stand.

Slip a loop of cord over the beam and observe the balance reading $f$ when the beam is horizontal. Attach a weight of 5 kilos to the loop and measure the horizontal distance $x$ of the loop from the knife edge. Read the balance indication F.

Repeat the observations with six different values of $x$, and tabulate them. The distance (X) between the knife edge and the vertical through the spring balance to be kept constant, also the load of 5 kilos.

Add to your table two columns, one containing the product $5x$ and the other $(F - f)X$.

Find the sum of each of these two columns.

Give a graph coordinating F and $x$. Select some value of $x$ not previously observed, and find from the graph the corresponding value of F. Also find F by adjusting the loop at a distance $x$ from the knife edge and reading the spring balance. (Army.)

**2.** Make a hole in the middle of the longer edge AB of a rectangular piece of cardboard. Let the cardboard hang from a knitting needle passed through this hole.

Suspend a 20 gram wt. (say) by a piece of cotton tied to B, so that the rectangle turns through an angle of about 50°. From measurements taken on the cardboard, find its weight.

**3.** Given an irregular piece of stout cardboard with six holes pierced through it.

Weigh the cardboard, and support it by a horizontal knitting needle passed through one of the holes, with a weight of 100 grams hung from another hole. Mark the point of suspension and the line of action of the 100 grams wt.

Transfer the 100 grams wt. to another hole, and mark on the cardboard its new line of action.

From these two observations determine and show on the cardboard its C. of G.

Check your result by letting the cardboard hang freely, and show on it the line which is then vertical.

**4.** The end A of the given swinging rod AB is tied to the support C by a thread AC. Fasten a spring balance to the other end B by another thread BD.

Measure the depth of A below C.

Lift the end B by the spring balance, so that AB is horizontal and the string vertical.

Read the spring balance.

Draw the end A of the rod slightly to one side by a thread from A to a clip at E.

Measure the vertical depth $y$ of A below C.

Again, bring AB to a horizontal position with the spring balance, keeping the thread BD parallel to AC. State how you do this.

Record T, the reading of the spring balance.

Repeat the observations, varying the length of AE, and always keeping BD parallel to AC.

Tabulate three values of T and $y$.

Explain your result theoretically. (Army.)

**5.** Arrange the bar ABC and the metre rule DEF as shown, so that when both are held horizontally the three threads AH, CD, EK may be all vertical. D and E are loops of thread tied tightly round the rule at graduations 10 cm. and 25 cm. respectively. From B suspend the load W, and note the distance $y$ cm. from E of the rider R when placed so as to balance the system. Tabulate $y$ when W = 0, 100, 200, 300, 400, 500 grams.

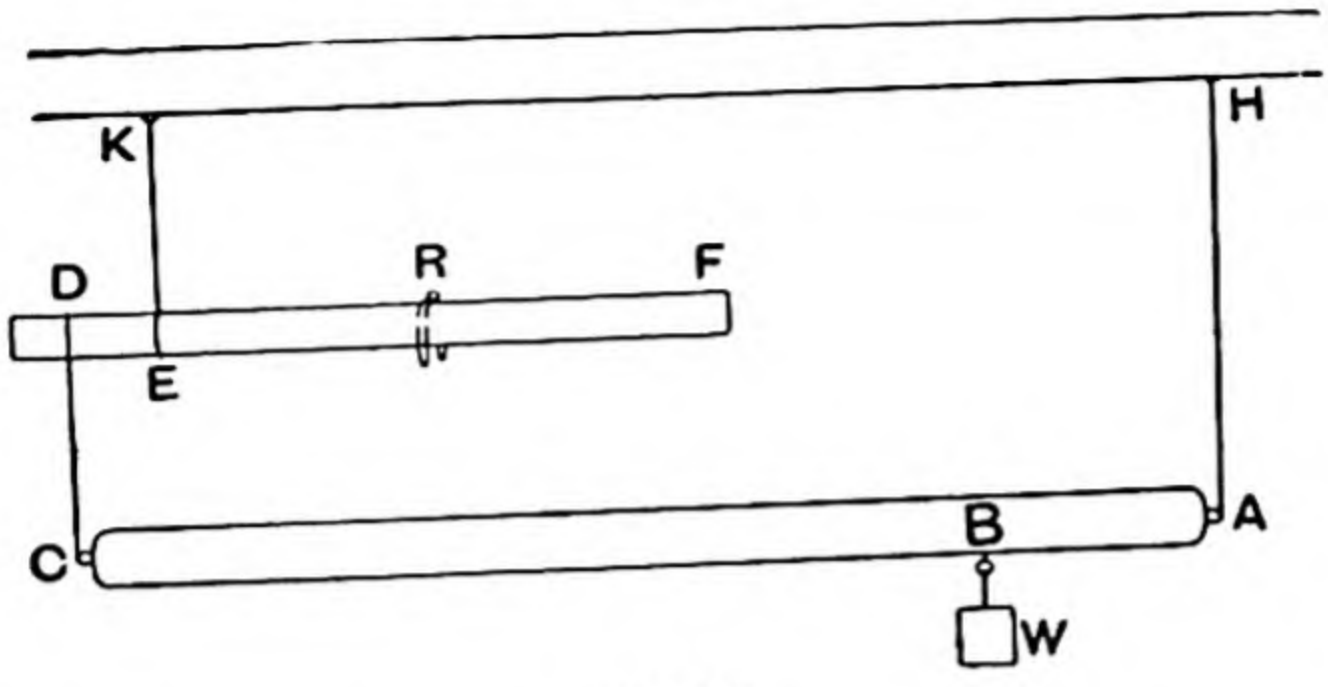


FIG. 34.

Show in a graph how $y$ depends upon W, and account for the fact that the graph is a straight line. (Army.)

## CHAPTER II.

### MACHINES.

**Work.**

We are accustomed to use the phrase 'doing work' to express the fact that we are engaged in some kind of laborious effort. For instance, we should say that we are doing work if we raised a weight of 10 lbs. through a height of 5 feet. We should also probably say we were doing work if we pushed with considerable force against a wall, without producing any effect on the wall, but, in the scientific sense of the phrase, the force in this case is doing no work. A force is said to do work when its point of application is displaced in the direction of the force. The amount of work done is measured by the product—**Force × distance the point of application moves in the direction of the force's line of action.**

Work may be measured in foot-pounds, foot-tons, etc. If a weight of 10 lbs. is slowly lifted vertically through a height of 5 ft., the work done against gravity is 50 ft.-pounds. It requires a knowledge of dynamical principles to show that work must be done to increase the velocity with which a body moves, but if the body is raised slowly the work done in setting it in motion may be neglected.

An important dynamical principle, which may be proved experimentally, states that, if the *rate of change of the*

*velocity* of a body is zero, then the resultant force acting on the body is zero, so that the relation between the forces acting on a body *moving with uniform velocity*, is the same as the relation between these forces when they act on a body at rest.

Any relations that we determine between forces in equilibrium acting on a rigid body at rest will be true of forces acting on a body moving with uniform velocity.

It will be noted that the time taken does not affect the question. The body may be raised in small portions at a time, but when the 5 lbs. have been raised through 4 ft., 20 ft.-lbs. of work will have been done in all.

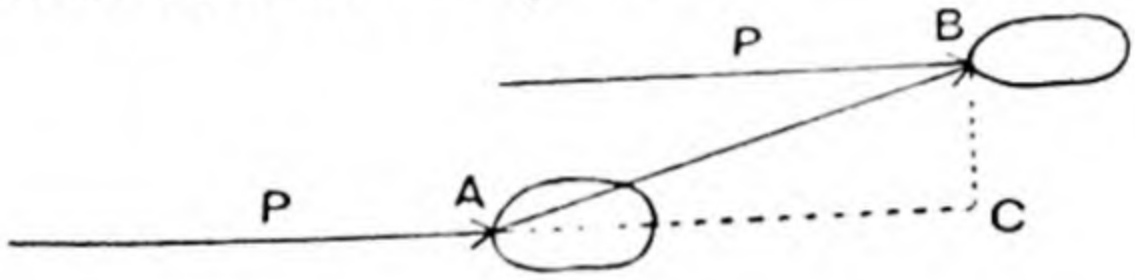


FIG. 35.

If the point of application of the force P, which acts on the body at A, is moved from A to B, the force remaining parallel to its original direction, the work done by the force P will be P × AC, and not P × AB. It is clear that another cause must have been responsible for moving the body in the direction CB.

If a weight is supported by a string, and moved horizontally, no work will be done against gravity, since the force of gravity acts at right angles to the direction of motion.

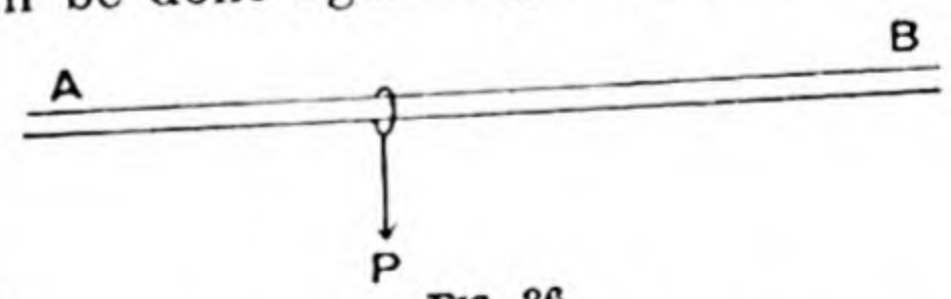


FIG. 36.

If a ring on a rough bar AB is pulled by a force P in a direction at right angles to the bar, no work will be done.

If the force P acts obliquely, and the ring is moved from C to D, the work done by P will be P × CE, where DE is

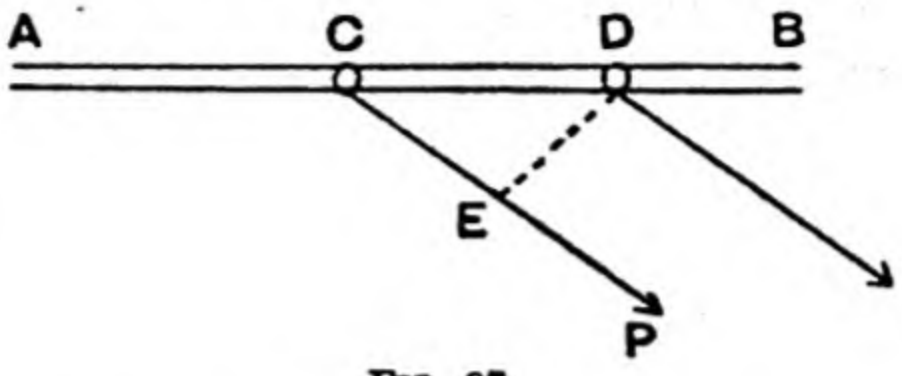


FIG. 37.

drawn at right angles to the direction of P, since CE is the distance the point of application has moved in the direction of the force.

This result will be discussed at a later stage (see p. 85).

**Work done by a Lever.**

AOB is a lever balanced at O, its C. of G., and a weight $W_1$ at A balances a weight $W_2$ at B.

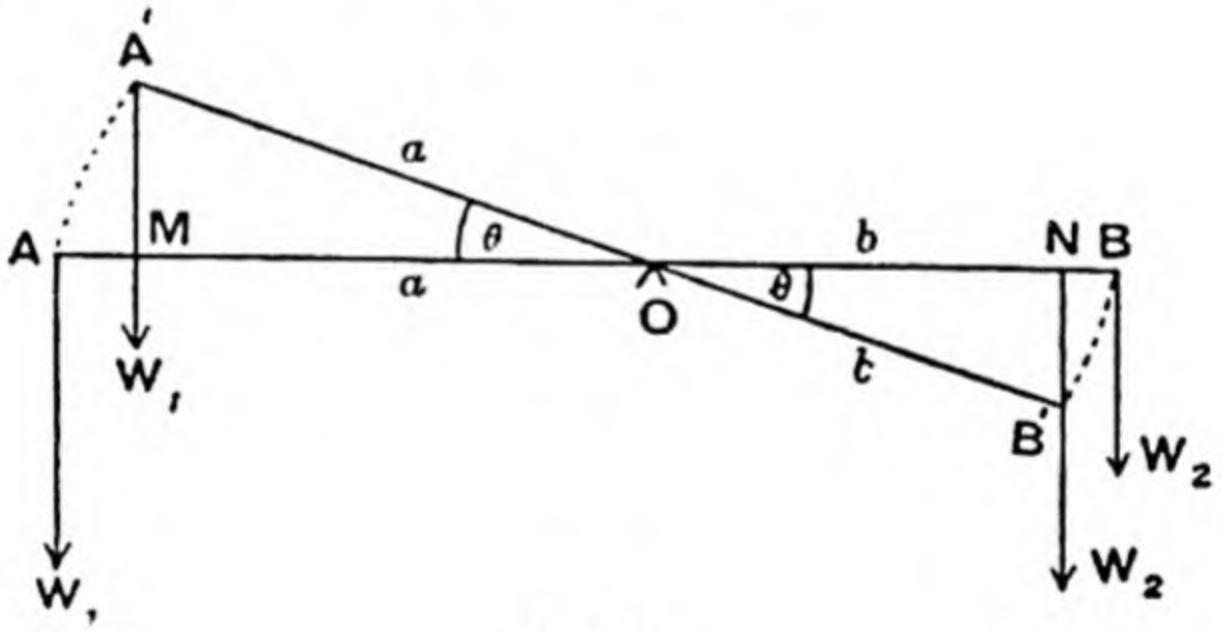


FIG. 38.

If the lever is turned through an angle $\theta$, will it balance in the new position? The distance moved by $W_1$ in the vertical direction in which it acts, is $a \sin \theta$, so that, according to our definition of work, the work done on $W_1$ is $W_1 a \sin \theta$.

Similarly, the work done by $W_2$ is $W_2 b \sin\theta$.

But, by the principle of the lever,

$$W_1 a = W_2 b\,;$$

$$\therefore\ W_1 a \sin\theta = W_2 b \sin\theta,$$

*i.e.* the work done on the machine is equal to the work done by it.

This is only true in an ideal machine, as we neglect the work done in turning the lever round O against the friction at the axis, and also the work required to set the weights and lever in motion.

The **efficiency** of a machine is measured by the ratio

$$\frac{\text{Useful work done by the machine}}{\text{Total work done on the machine}},$$

so that in an ideal machine the efficiency is unity or 100 %.

The principle of work, as applied to machines, was first discussed by Stevinus (1548-1620) (see p. 12), who stated it thus: "Ut spatium agentis ad spatium patientis, sic potentia patientis ad potentiam agentis."

*i.e.* $$\frac{\text{Distance moved by the effort}}{\text{Distance moved by load}} = \frac{\text{load}}{\text{effort}},$$

The principle was applied to the inclined plane by Galileo.

## EXAMPLES V.

**1.** If 1 ton of coal is raised to the top of a pit, 500 ft. deep, find the work done against gravity in ft.-tons and in ft.-lbs.

**2.** What force will be required to raise a weight of 500 grams, at a uniform velocity of 20 cms. per second? Find how much work is done in raising this weight through 1 metre. (Answer in cm.-grams.)

**3.** The resistance to the motion of a bicyclist on a level road being a force of 6 lbs. wt., find the work in ft.-tons which he does against this resistance in riding 10 miles.

**4.** A weight of 1 ton is to be raised 20 ft. by a machine; find how much work must be done by it. If its efficiency is 60 %, find how much work must be done on the machine. How much work is wasted ?

**5.** Two strings are so connected that when one is pulled down 3 ft. the other rises 1 ft. If you hang a weight of 5 lbs. on the former, what is the least pull you must exert on the latter to raise it ? Give your reasons.

**6.** A body weighing 3 tons is kept moving with uniform velocity up a slope against a frictional resistance of 10 lbs. wt. per ton. Find the work done against friction in moving the body 20 ft.

**7.** When one of the keys of a pianoforte is depressed through $\frac{2}{5}$ in., the hammer of the pianoforte is raised through a vertical height of 2 in. A force of 2 oz. wt. will just depress the key. Find the weight of the hammer. (Neglect all weights except the hammer.)

**8.** A force of 10 lbs. wt. at one end of a lever 4 ft. from the fulcrum is required to raise a weight of 35 lbs. at the other end, which is 1 ft. from the fulcrum. Find the efficiency of the machine.

**9.** The efficiency of a lever is 80 %. Find what weight, 8 in. from the fulcrum, can be raised by a force of 20 lbs. wt. acting 3 ft. from the fulcrum.

**10.** A man pulls a rope through 40 ft. with a force of 50 lbs. wt., in order to raise a weight of $\frac{1}{5}$ ton. If the efficiency of the machine is 80 %, find how far the load will be raised.

**Pulleys.**

*Experiment* 1. Fasten the ends of a string to two equal weights, and put them over a pulley. They will be found to balance one another. We shall find, however, that we can add a small weight to either side without disturbing the equilibrium, so that the pulley does not turn when the weights are slightly unequal.

If this pulley were of larger radius, would the additional weight necessary to produce rotation be increased or diminished ? Why ?

Explain the advantages of large pulleys, such as those over which the wire rope passes to raise the cage at a coal pit.

If we take a pulley, carefully mounted on bearings, we shall find that a very small difference in the weights will produce motion, and that the difference depends chiefly on the friction at the axle.

Consider the equilibrium of the weights $W_1$ and $W_2$, and state the value of $T_1$ and of $T_2$, the tension in the two parts of the string.

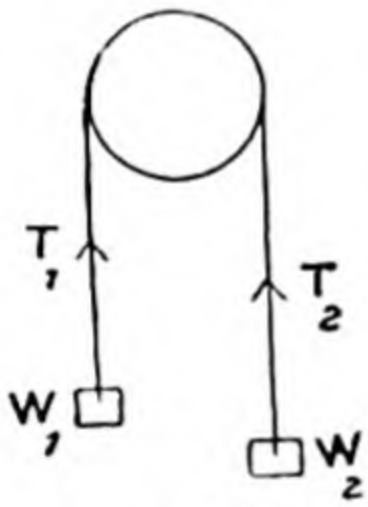


FIG. 39.

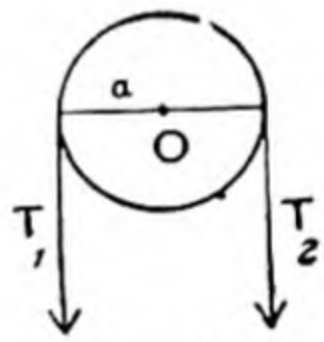


FIG. 40.

Considering the equilibrium of the pulley we have, by taking moments about O, if $a$ is the radius of the pulley :

$T_2a$, the moment of the force, tending to turn the pulley clockwise.

$T_1a$, the moment of the force, tending to turn the pulley in the opposite sense.

The difference between these represents the moment of the frictional force at the axis, and it is only when we assume that this force is negligible that we get $T_2a = T_1a$,

or $T_2 = T_1$, and can state in this case that the tension of the string is unaltered by passing round the pulley.

What is the force on the axle of the pulley ?

Find, by experiment, whether the friction at the axle is increased by increasing the weights which it is supporting.

*Experiment* 2. Pass a string round a pulley, fasten each end to a dynamometer, and hang a weight W to the pulley.

If T is the tension in the string, and $w$ is the weight of the pulley, we have, since the pulley is in equilibrium, $2T = w + W$.

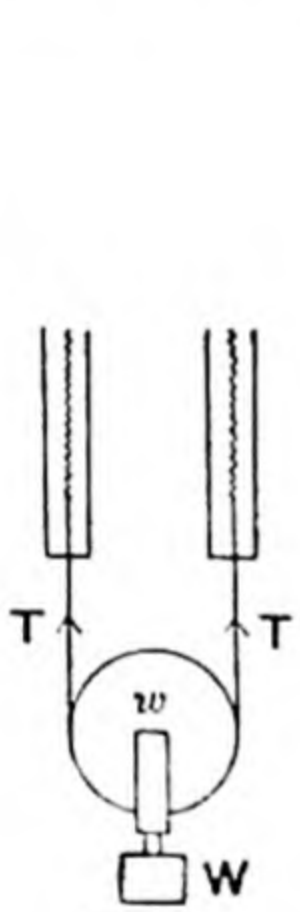


FIG. 41.

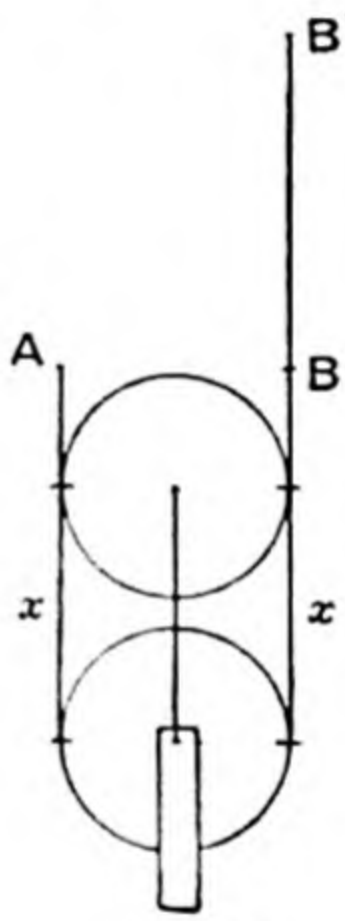


FIG 42.

If one end A of the string is kept fixed, and the other end B is raised until the pulley's centre has risen a distance $x$, what is the distance BB′ through which the other end B of the string has been raised ?

**System of Pulleys with a Single String. (Block and Tackle.)**

This system consists of two blocks containing the same number of pulleys, called sheaves. The upper block is fastened to a beam, the lower supports the load.

One end of a string is fastened to the bottom of the upper block, passes round a sheave of the lower block, then round a sheave in the upper block, and so on.

The force (P) required to raise the load (W) is applied at the other end of the string. It is called the *Effort*.

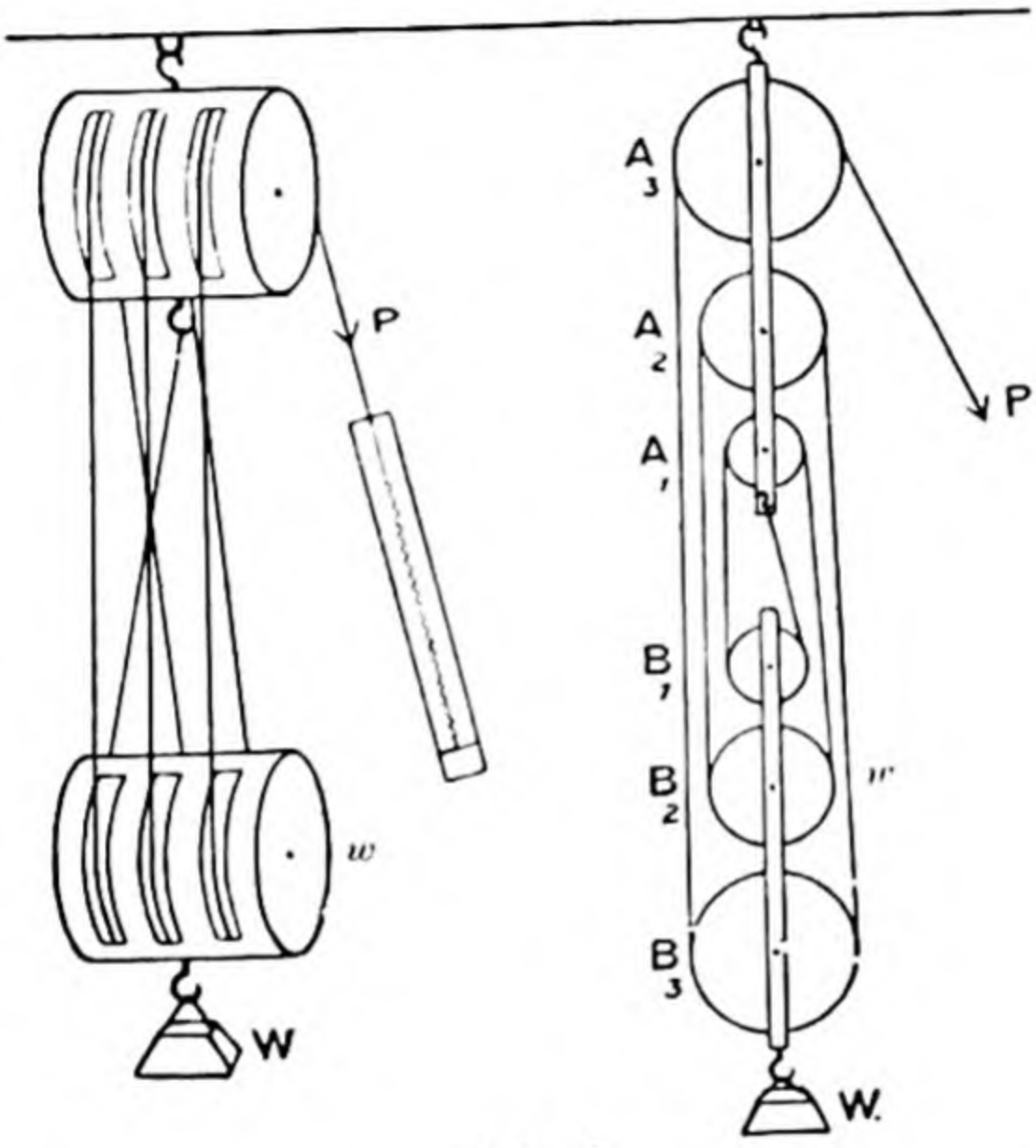


FIG. 43.

The model on the right shows the pulleys placed beneath one another, so that the arrangement of the string is more easily seen.

If the load W and the whole of the lower block is raised a distance $x$, the pulley $B_1$ will rise a distance $x$, and a length of string $2x$ will pass round $A_1$. Since $B_2$ has risen a distance $x$, a length $4x$ will pass round $B_2$, and $6x$ round

$B_3$ and $A_3$, so that the effort P will move a distance $6x$. The fraction $\frac{\text{distance P moves}}{\text{distance W moves}}$ is called the *Velocity Ratio* of the machine.

The useful work done by the machine is $Wx$, if the load rises a distance $x$. The work done on the machine is $P \times 6x$.

$$\text{efficiency of machine} = \frac{W}{6P}.$$

The total work done will be $(W + w)x$, if no work were used in overcoming friction, and in this case we should have

$$(W + w)x = P \,.\, 6x,$$

or
$$W + w = 6P.$$

Consider now the equilibrium of the lower block. If we neglect friction, the tension in the string will be the same throughout and equal to P. Therefore, if we assume the strings are parallel, the total force acting vertically upwards on the block will be 6P, since there are 6 portions of the string; hence $6P = W + w$, as before.

The fraction $\frac{\text{load}}{\text{effort}} = \frac{W}{P}$ is called the *Force Ratio* of the machine.

*Experiment* 3. If scale pans are used, it is best to take the end of the string over another pulley fixed to the beam, so that the scale pan containing the effort hangs clear of the lower block. The weight of the scale pans may be included as part of the load and effort respectively.

In an experiment with this system of pulleys, the following results were obtained :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Load, - | 230 | 260 | 300 | 360 | 430 | 505 | 560 | 660 grams. |
| Effort, - | 60 | 66 | 75 | 86 | 100 | 115 | 125 | 145 grams. |

In each case the effort just caused the load to ascend.

**Efficiency.**

For a load (W) of 230 grams the effort (P) is 60 grams, and when W ascends 1, P descends 6; therefore the efficiency for this load is

$$\frac{230 \times 1}{60 \times 6} = \cdot 64 \text{ or } 64\ \%.$$

Calculating the efficiency in each case we get

| Load, - | 230 | 260 | 300 | 360 | 430 | 505 | 560 | 660 grams. |
|---|---|---|---|---|---|---|---|---|
| Efficiency, | 64 | 65·8 | 66·6 | 69·8 | 71·7 | 73·2 | 74·6 | 75·8 % |

How do you account for the fact that the efficiency increases when the load is increased?

**Law of the Machine.**

Plot out the corresponding values of P and W on a graph: P along the Y axis and W along the X axis. We find that

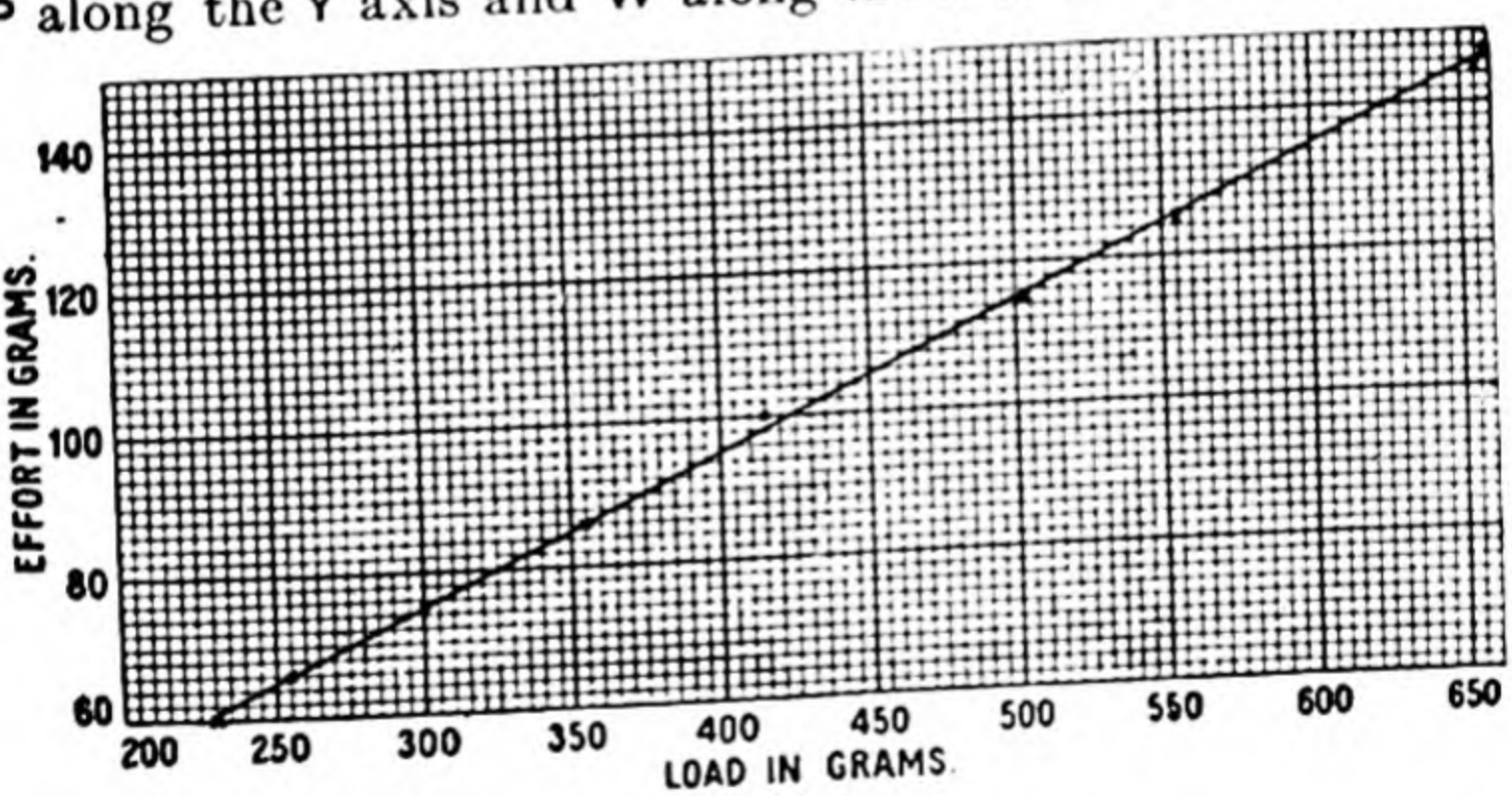


FIG. 44.

the points lie approximately along a straight line. Draw the line so that there are as many points on one side of it as on the other.



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The equation of a straight line is of the form $y=ax+b$ (see Algebra), in this case $P=aW+b$.

To find $a$ and $b$ we select two points on the graph, *e.g.* $P=110$, $W=475$ and $P=125$, $W=550$; then $110=475a+b$ and $125=550a+b$, from which we obtain $a=0{\cdot}2$, $b=15$; hence $P=0{\cdot}2W+15$, which is the Law of the Machine.

**Friction.**

In the experiment referred to above, the weight of the lower block was 73 grams, so that for a load of 300 grams the *total work* done against gravity when the load is raised a distance $x$ is $373x$ units. The effort was 75 grams,

$\therefore$ the work done on the machine $=75\times 6x=450x$ units,

*i.e.* $77x$ units of work were used to overcome friction. This requires a force of $\frac{77x}{6x}=13$ grams as part of the effort required to overcome the friction of the machine.

Calculate how much of the effort is required to overcome the friction of the machine for each load, and verify the following table:

| Load, - | 230 | 260 | 300 | 360 | 430 | 500 | 560 | 660 |
|---|---|---|---|---|---|---|---|---|
| Friction - | 10 | 10·5 | 13 | 14 | 16 | 19 | 20 | 23 |

How do you account for the fact that the friction increases when the load increases?

*Experiment* 4. We can find the effort required to overcome friction in a different way.

Let P be the effort required to raise W 300 grams, if there had been no friction; let F be the effort required to overcome friction.

In order to raise W with uniform velocity we require an effort of 75 grams. $\therefore P+F=75$. ..... .....(1)

Now find what effort is required just to keep W falling with uniform velocity. With the same apparatus it was found to be 50 grams.

In this case the effort is less than P, by the frictional force ;

$$\therefore\ P - F = 50. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting (1) and (2), we get

$$2F = 25\ ;$$

$$\therefore\ F = 12{\cdot}5 \text{ grams,}$$

approximately the same result as before.

This also gives $P = 62{\cdot}5$, and since the total load raised is $300 + 73 = 373$ grams, we see that approximately $6P = W + w$, which is the relation when friction is ignored.

## EXAMPLES VI.

**1.** In a system of pulleys the velocity ratio is 8. If a load of 1 ton is raised by an effort of 3 cwt., find the efficiency of the machine, and state the force ratio.

**2.** If 20 % of the work done on a machine is wasted, find its efficiency.

**3.** The efficiency of a machine is 70 %. A load of 1 cwt. is raised by an effort of 20 lbs. ; find the distance the load rises if the effort moves through 5 feet.

**4.** The velocity ratio of a machine is 4 and its efficiency is 60 %. What load can be raised by an effort of 1 cwt. ?

**5.** In a machine a pull of 80 lbs. will raise a load of 2 cwts. If the effort moves through 5 ft. while the load is rising 1 ft., find the efficiency of the machine.

**6.** A block and tackle has two sheaves in each block. If the block weighs 5 lbs. and 10 % of the work done is used in overcoming friction, find what effort is required to raise 5 cwt., and find the efficiency of the machine.

**7.** A block and tackle has three sheaves in each block.

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The following table gives the effort required to raise the given loads:

| Load, - | 20 | 50 | 100 | 150 | 200 lbs. |
|---|---|---|---|---|---|
| Effort, - | 4·9 | 10·0 | 18·6 | 27·3 | 36·2 lbs. |

Find the efficiency in each case and find the force required to overcome friction when the load is 100, if the weight of the lower block is 3 lbs.

**8.** The following table gives the effort P kg. required to raise W kg.:

| W, - | 50 | 100 | 200 | 300 | 400 | 500 |
|---|---|---|---|---|---|---|
| P, - | 106 | 122 | 152 | 184 | 216 | 246 |

Plot these values, and, assuming the relation between P and W to be of the form $P = aW + b$, find the values of $a$ and $b$.

Find the efficiency of the machine for a load of 250 kg. if the velocity ratio is 4.

**9.** A block and tackle has three sheaves in each block. When a rope passes over a pulley the tension on one side is $\frac{7}{8}$ of its value on the other. If P is the effort, find the tension in the part of the rope which is fastened to the upper block, (i) when the load is ascending, (ii) when the load is descending.

**10.** In the system of pulleys with a single string a force of 50 lbs. wt. supports a load of 220 lbs., and a force of 25 lbs. wt. supports a load of 95 lbs. What force will be required to support a load of 315 lbs., and what is the weight of the lower block? Friction may be neglected.

**11.** In a block and tackle with three sheaves in each block, the effort required to raise a load of 465 lbs. was 111·5 lbs. When the effort was reduced to 68·5 lbs., the load just ran down. Find how much of the effort was used in overcoming friction. Find, by the principle of work, the weight of the lower block.

**12.** If the load raised by the system in Qu. 9 is W, and the weight of the lower block is $w$, prove that the efficiency of the machine is $\frac{7W(1-(\frac{7}{8})^6)}{6(W+w)}$.

**Archimedes' System.**

In this system we have one fixed pulley and a series of movable pulleys, each having a separate string, one end of which is fastened to the beam, and the other end to the next pulley above.

If the load ascends a distance $x$, prove that when there are three movable pulleys the effort will descend a distance $8x$.

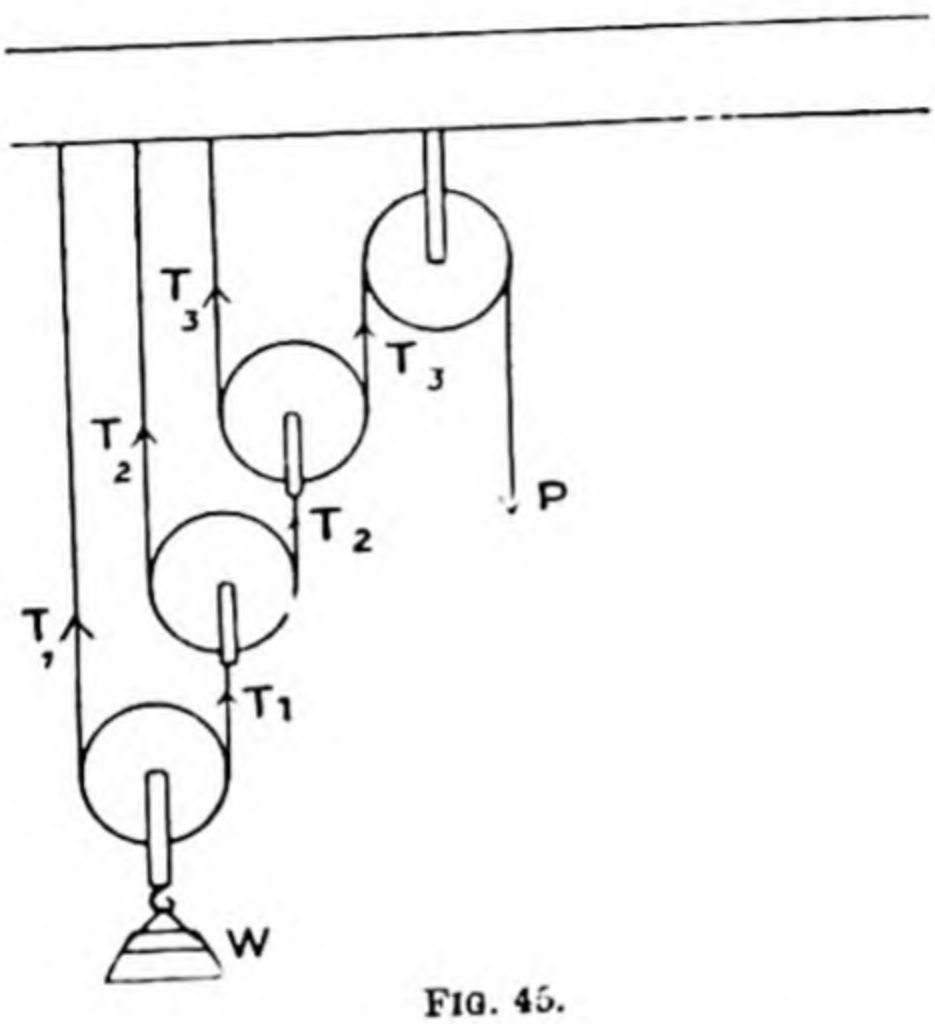



FIG. 45.

If an effort P just lifts a load W, we have efficiency $= \frac{W}{8P}$.

If the weight of each pulley is $w$, the total work done, excluding friction, is

$$Wx + w \cdot x + w \cdot 2x + w \cdot 4x = x(W + 7w).$$

Neglecting friction,

$$P(8x) = (W + 7w)x,$$

or

$$8P = W + 7w.$$

The portion of the effort required to overcome friction may be found in the same manner as in the last system.

The relation between the effort and the load may be found by considering the equilibrium of each pulley separately, if we neglect friction.

For the bottom pulley $2T_1 = W + w$.

For the next pulley, which is pulled downwards by the string whose tension is $T_1$, we have

$$2T_2 = T_1 + w.$$

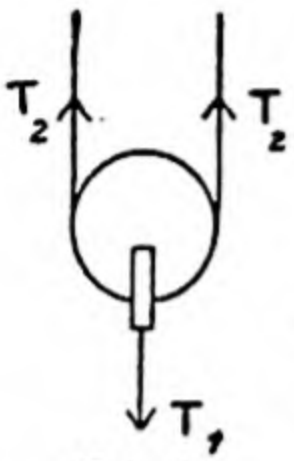


FIG. 46.

Similarly, $2T_3 = T_2 + w$

and $T_3 = P$.

Hence $8P = W + 7w$.

The following results were obtained from a system of this kind with three movable pulleys:

| | | | | | |
|---|---|---|---|---|---|
| Load W, - | 63 | 113 | 153 | 213 | 263 grams. |
| Effort P, - | 32 | 39 | 46 | 55 | 64 grams. |

Calculate the efficiency in each case.

The weight of each movable pulley was $w = 20$ grams.

Calculate the effort required to overcome friction for each load.

The effort required when 113 was on the point of descending was 24. Hence find the frictional force for this load.

## EXAMPLES VII.

**1.** An ordinary block and tackle has two pulleys in each block. If the system were frictionless and the pulleys of negligible weight, what force would have to be exerted to lift a load of 400 lbs.? If the efficiency of the system is actually 0·4, find the force required to lift a load of 400 lbs.

**2.** The law connecting the effort P lbs. and the load W lbs. in a lifting tackle is given by the equation $P = 0{\cdot}052W + 2{\cdot}4$. Plot the graph of this equation, and read off the value of P when $W = 140$ and 460. Find the efficiency in each case if the velocity ratio is 20.

**3.** In a pulley tackle the velocity ratio is 8, and the friction of the tackle is such that only 55 % of the effort applied is usefully employed in lifting the weight. Find what effort would have to be exerted to raise a load of $\frac{1}{4}$ ton.

**4.** Find the velocity ratio of the system shown in Fig. 47. It is found that a force of 27 lbs. raises a load of 10 lbs., and that a force of 78 lbs. raises a load of 80 lbs. Find the efficiency in each case. If $P = aW + b$, find the law of the machine.

**5.** Find the velocity ratio of the system of pulleys known as the Barton (see Fig. 48).

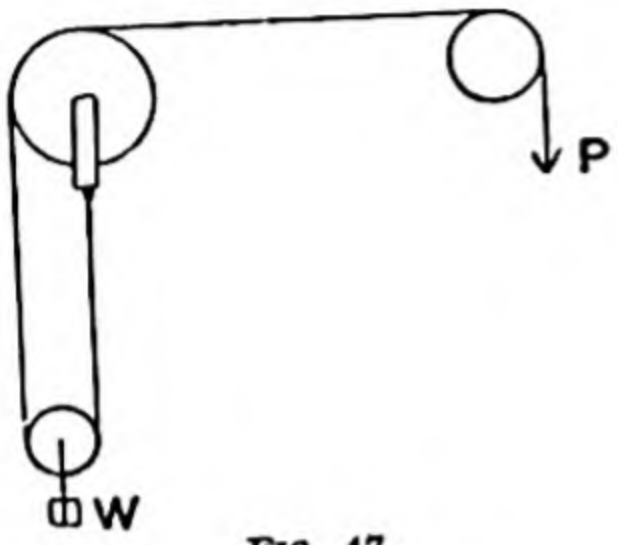



FIG. 47.

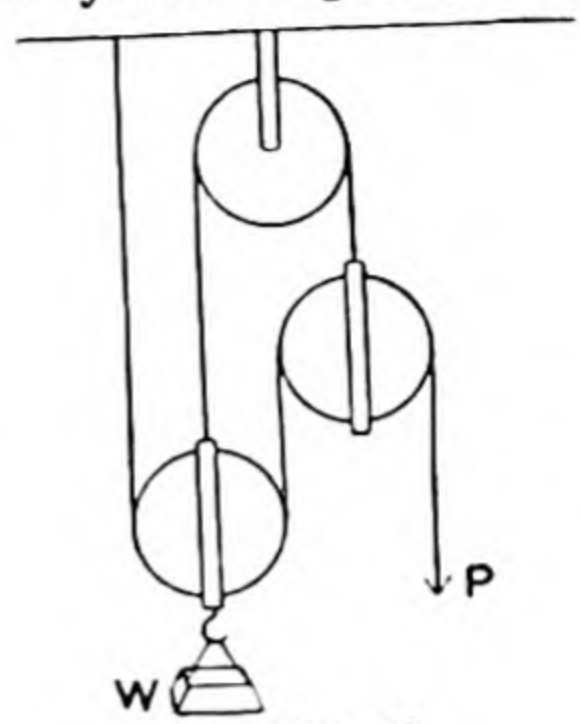



FIG. 48.

If the weight of the movable pulleys is 10 lbs. each, find what effort will raise a load of 200 lbs., neglecting friction.

If an effort of 50 lbs. raises a load of 1 cwt., find the efficiency of the system.

**6.** A machine is constructed so that a load of W lbs. can just be lifted by an effort of P lbs., where $P = 3{\cdot}7 + 0\ 058W$, and it is noticed that P must move through 30 ft. to raise W through 1 ft. Find the efficiency of the machine when $W = 100$ and when $W = 1000$.

**7.** A block and tackle has two pulleys in each block. From the movable block is suspended a basket containing a man, the weight of the man and basket being W. If the man pulls the free end of the rope, what force must he exert to raise himself if the weight of the lower block and friction were neglected?

**8.** The following table gives the relation between the load and the effort in the Archimedes system with three movable pulleys:

| | | | | | |
|---|---|---|---|---|---|
| Load, - | 262 | 362 | 412 | 462 | 512 grams. |
| Effort, - | 62 | 74 | 82 | 89 | 97 grams. |

Find the efficiency in each case and the Law of the Machine. Calculate how much of the effort was due to friction when the load was 412 grams, if each pulley weighed 20 grams.

**The Weston Differential Pulley.**

The upper block A of this system has two grooves side by side, one of which has a slightly greater radius than the

FIG. 49.

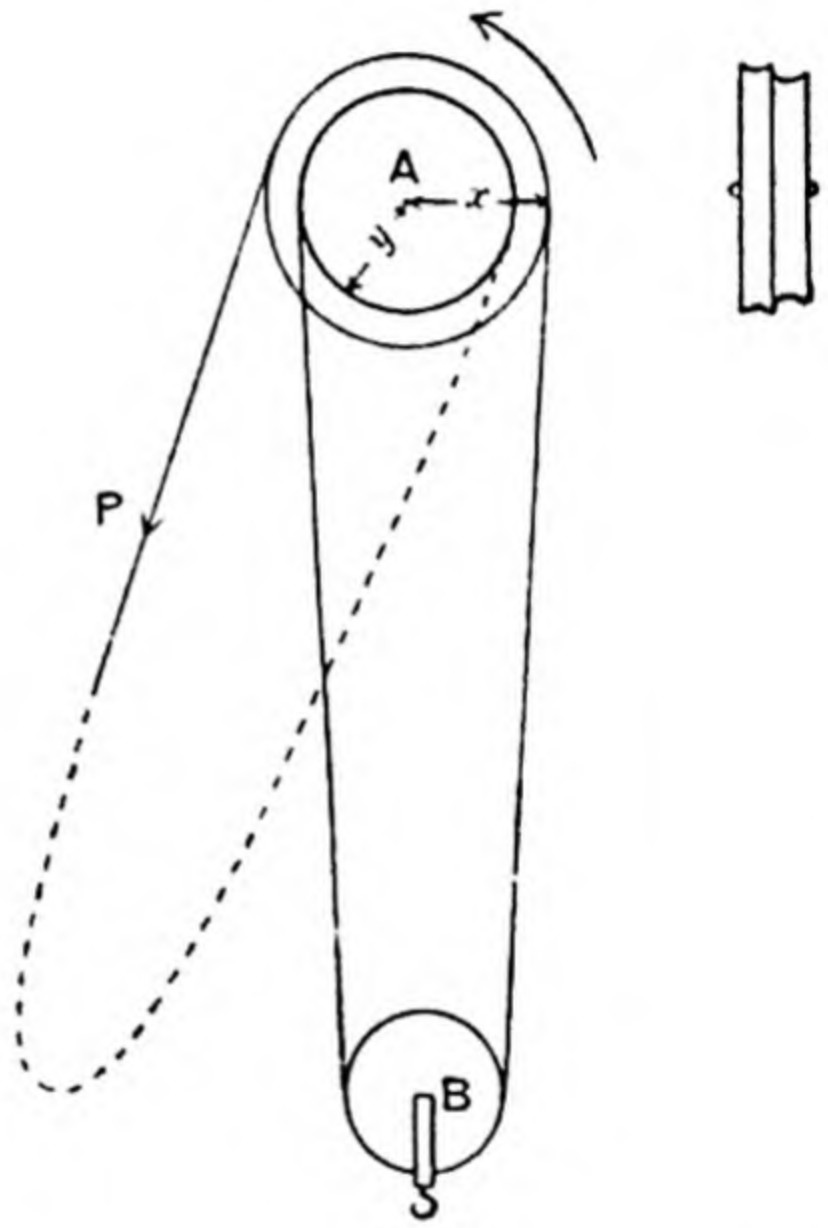



FIG. 50.

other. The rims are recessed so that the chain cannot slip round them.

An endless chain passes round the rims of the upper block and round a pulley B, in the manner shown by the figure.

Suppose the greater radius of the upper block to be $x$ and the smaller $y$; then, if the block be turned through a radian, a length $x$ of chain will be wound up, and at the same time a length $y$ will be let down.

$\therefore$ altogether a length $x-y$ will be wound up.

$\therefore$ the lower pulley rises a distance $\frac{x-y}{2}$.

Meanwhile P descends a distance $x$.

$\therefore$ work done on the machine $= Px$.

Useful work done by the machine $= \frac{W(x-y)}{2}$;

$$\therefore \text{ efficiency} = \frac{W(x-y)}{2Px},$$

and
$$\text{velocity ratio} = \frac{x}{\frac{x-y}{2}} = \frac{2x}{x-y}.$$

If $x=7$ in. and $y=6\frac{3}{4}$ in., this becomes $\frac{14}{\frac{1}{4}}=56$, so that P must descend 56 in. for W to rise 1 in.

If $e$ is the efficiency, we have

$$P = \frac{W(x-y)}{2xe}.$$

Therefore, for a given load, P diminishes as $x-y$ decreases, so that $x$ and $y$ are made very nearly equal. Why are they not made exactly equal?

**Overhauling.**

When a machine is used for raising heavy weights it is important, in order to avoid accident, that when the effort is removed, the load should not run back or "take charge." If this should happen, the machine is said to overhaul.

It can be shown that if the efficiency of a machine is less than 50 %, it will not overhaul.

Suppose the work done by the effort to be 100 ft.-lbs. If the efficiency of the machine were 40 %, the useful work done would be 40 ft.-lbs., so that 60 ft.-lbs. would be required to overcome friction, etc. If the effort were removed, there would be some diminution of friction, but with heavy loads and in machines with a large velocity ratio, the effort is small compared with the load. It follows that, since nearly 60 ft.-lbs. of work must be done against friction, the work which the load can do in returning to its original position, *i.e.* 40 ft.-lbs., will be inadequate for the purpose, so that the load will be unable to run down.

The efficiency of the Weston pulley is usually less than 50 %, and its utility is due to this fact and to its large velocity ratio.

**Wheel and Axle.**

This machine, which is used for raising weights a considerable distance, consists of a wheel and an axle which

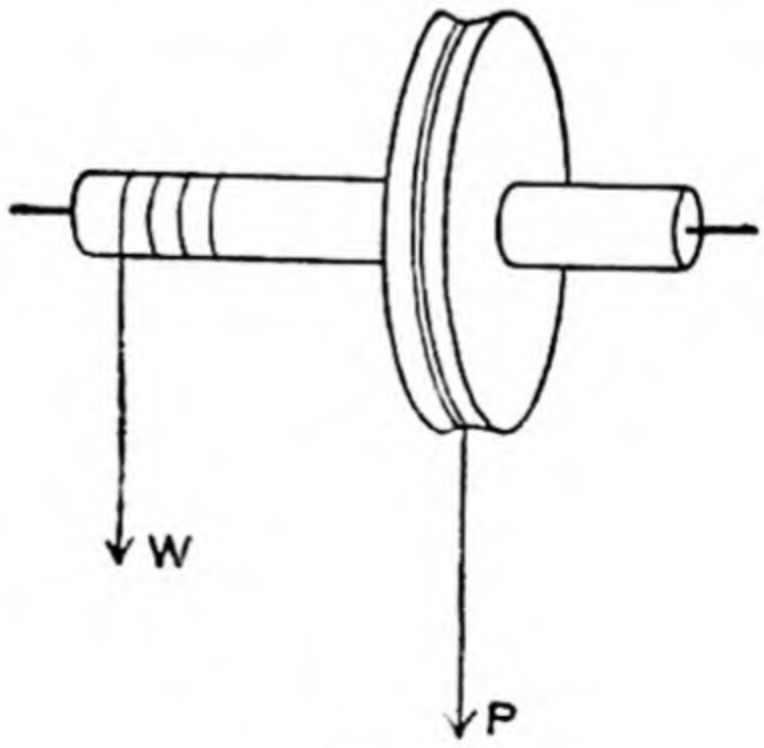



FIG. 61.

rotate together round the same axis. A force P, applied to a rope coiled round the wheel, raises a weight W fastened to a rope coiled in the opposite direction round an axle.

If the radius of the wheel is $a$ and the radius of the axle $b$, show that the efficiency of the machine is $\frac{Wb}{Pa}$.

**Differential Wheel and Axle.**

If a considerable weight is to be lifted by a small effort, the radius of the wheel would have to be inconveniently large, or the radius of the axle too small for safety. In this case the axle is made of two parts, having different radii, and the rope supporting the pulley to which the load is fastened, is wound up on the larger axle, while it unwinds on the smaller.

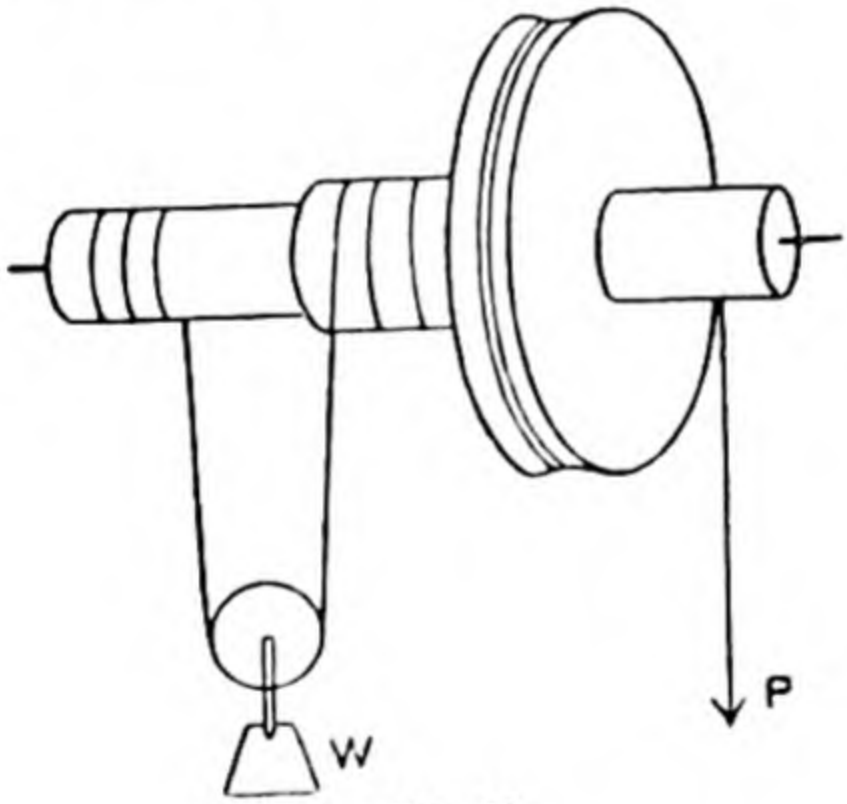



FIG. 52.

If the radius of the wheel is $a$, the radius of larger axle $b$ and the radius of the smaller axle $c$, prove tnat the efficiency of the machine is $\frac{W(b-c)}{2Pa}$.

**Inclined Plane.**

If a weight of P lbs. hanging vertically is required to support a load of W lbs., on an inclined plane whose length

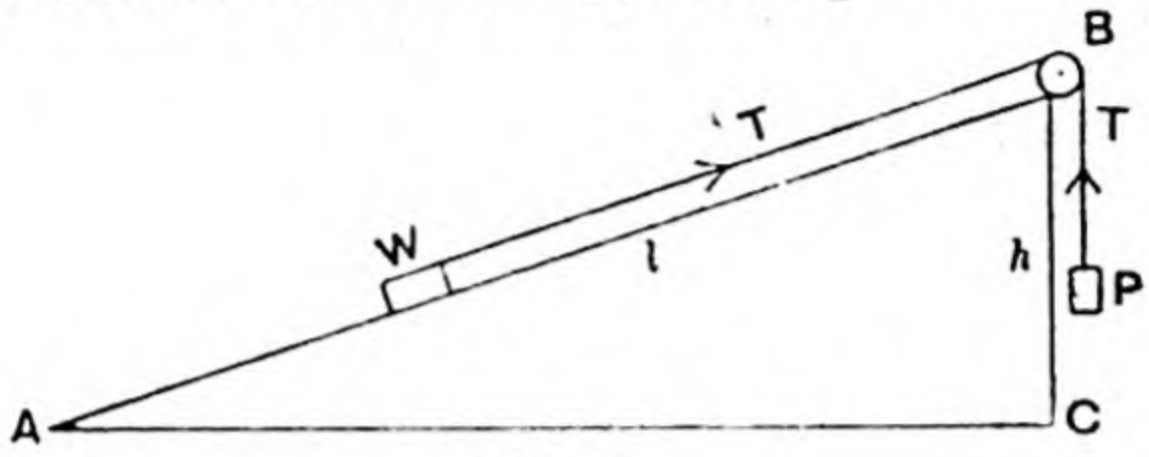



FIG. 53.

is $l$ and height $h$, then, if W is pulled the length of the plane, the distance moved by W in the vertical direction is $h$.

$\therefore$ work done on $W = Wh$.

Meanwhile P has descended a distance $l$.

$\therefore$ work done by $P = lP$.

If friction is neglected, we have

$$Wh = Pl \text{ or } \frac{W}{P} = \frac{l}{h}.$$

If the plane is smooth, there is no frictional force along its surface AB, and the pressure of the plane on the weight will be entirely at right angles to AB. Therefore this pressure does no work, since the motion of the body is always at right angles to the direction of this force.

## EXAMPLES VIII.

1. The radii of the sheaves in a Weston pulley block are 6 in. and 5½ in.; find what load could be supported by an effort of 20 lbs. if the efficiency is 45 %.

2. How many revolutions of the sheaves in Qu. 1 will be required to raise the load 1 ft.?

**3.** The radii of the sheaves in a Weston pulley block are 7 in. and 8 in.; find the efficiency of the machine if a load of 15 cwts. is raised by an effort of 2 cwt.

**4.** In a differential wheel and axle find what weight can be raised by an effort of 100 lbs. (neglecting friction), given radius of wheel 20 in. and radii of two parts of the axle 6 in. and 7 in.

**5.** The efficiency of a differential wheel and axle is 75 %. Find the effort applied tangentially to a handle 2 ft. long, measured from the axis, which will support a load of 2 cwt., given radii of the two parts of the axle 5 in. and 6 in.

**6.** Find what load can be supported on a smooth plane inclined at 40° to the horizontal by a force of 20 lbs. acting parallel to the line of greatest slope.

**7.** A weight of 2 cwt. is fastened to a rope which passes over a pulley at the top of an inclined plane, and supports a load on the plane. If the inclination of the plane to the horizontal is 20°, find the load supported if friction can be neglected.

**8.** The efficiency of a wheel and axle is 60 %. If the radius of the wheel is 1 metre, find how far an effort of 10 kg. will raise a load of 150 kg. in 5 complete turns.

**9.** In a Weston differential pulley it is noticed that the weight rises 1 ft. when 48 feet of chain are pulled through; if the radius of the greater sheave in the upper block is 6 in., find the radius of the other sheave.

**10.** Draw an end on view of a wheel and axle, and apply the principle of moments to find the relation between P and W, neglecting friction. Find, by the principle of moments, the relation between P and W in a differential wheel and axle.

### Cog Wheels.

If a pair of cog wheels are placed so that the teeth of one can act in the spaces between the teeth of the

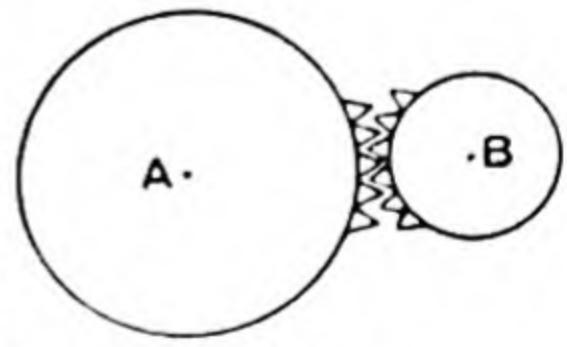



FIG. 54.

other, the wheels are said to be in gear, and if one is rotated the other will rotate in the opposite direction.

If A has 20 teeth and B has 10 teeth, then B will have made two revolutions when A has made one.

**The Winch.**

The double handle of the winch rotates with a small toothed wheel A, geared to a larger wheel B. This wheel rotates on the same axis as the barrel C, on which is coiled the rope raising the load.

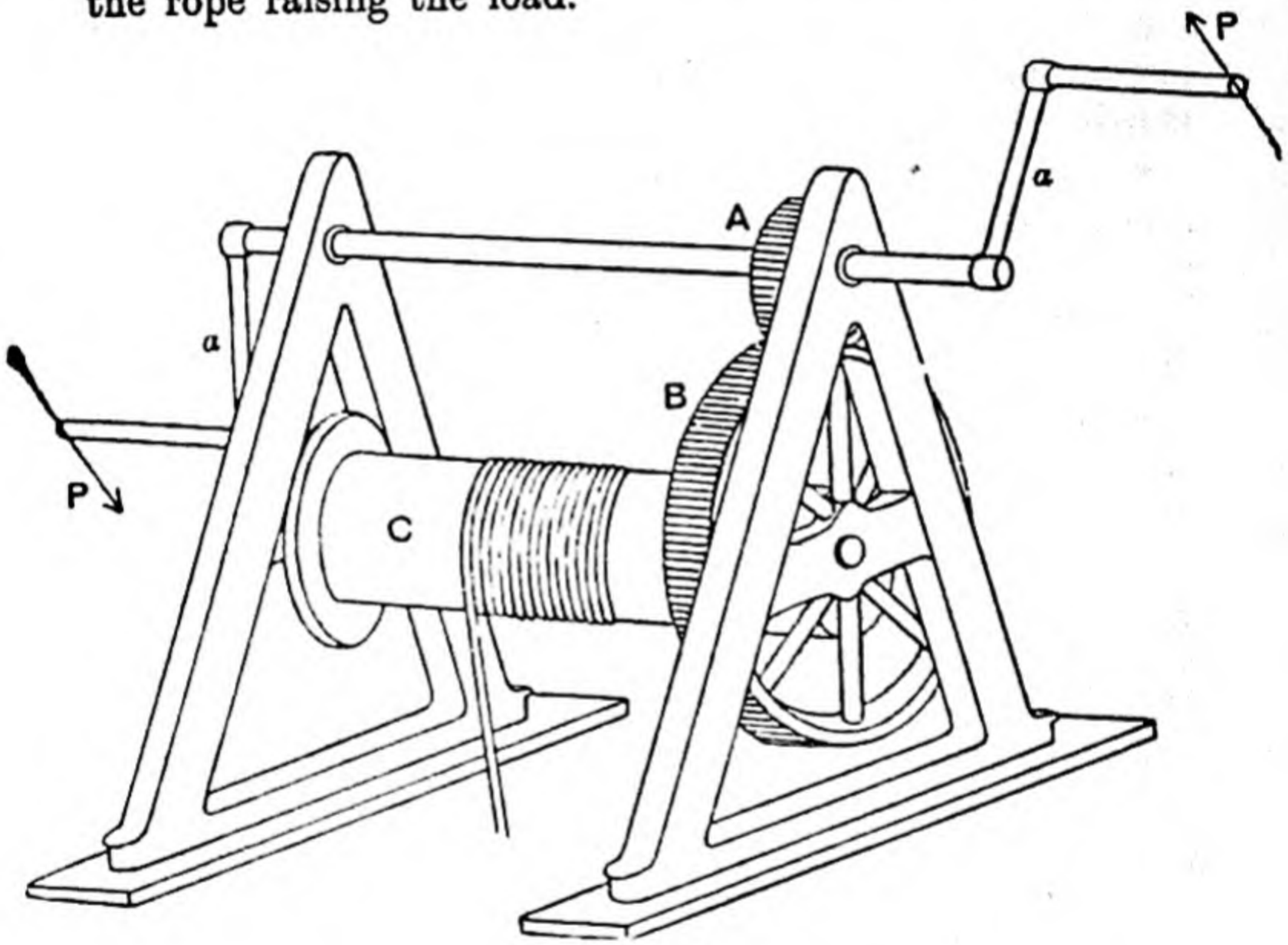



FIG. 55.

A force P acts tangentially to the path of each handle at the end of an arm whose length is $a$. If A has 10 teeth, B 40 teeth and the radius of the barrel is $b$, prove that the efficiency of the machine is $= \frac{Wb}{8aP}$.

**The Screw.**

On a rectangular sheet of paper ABCD, draw a diagonal AC. Fold the paper into a cylinder, so that CB coincides with DA. We now have a model of a screw, of which the line AC is the thread.

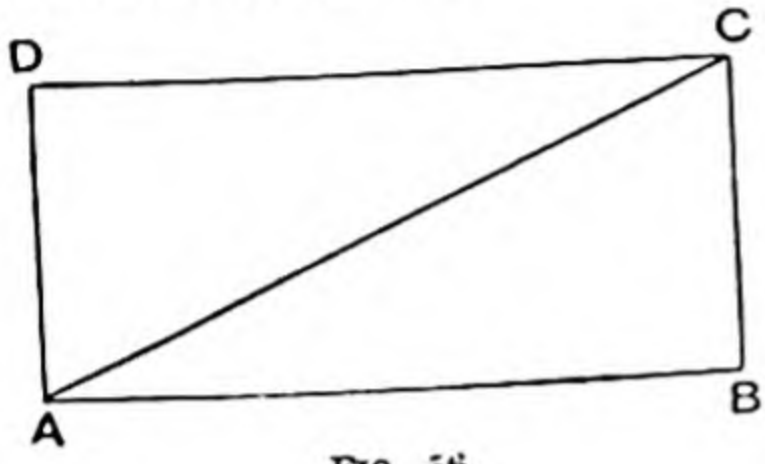



FIG. 56.

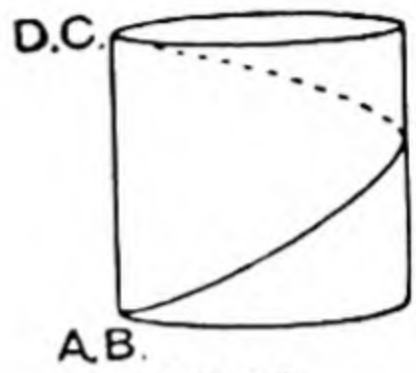



FIG. 57.

If the screw makes one revolution through the nut in which it works, it will be seen that the distance the screw moves parallel to its axis is equal to AB (Fig. 58). This distance between the threads, measured parallel to the axis of the screw, is called the pitch ($p$) of the screw.

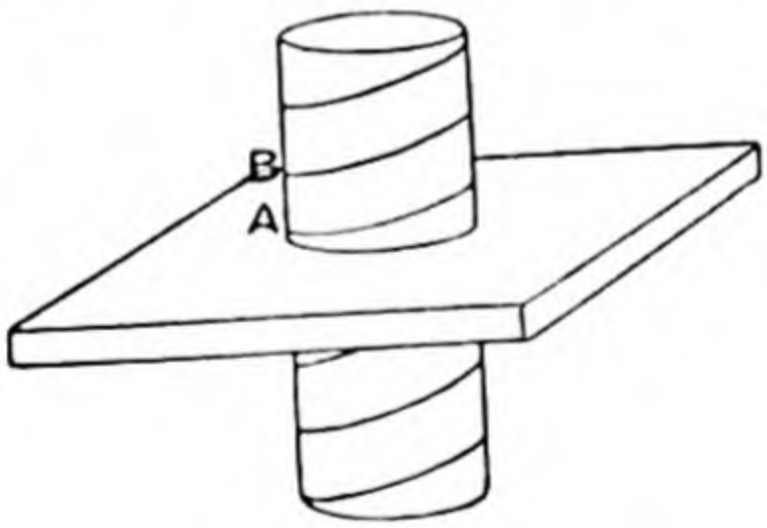



FIG. 58.

If a force P is applied at right angles to each end of an arm of length $a$, as in a letter press, the work done on the machine in one revolution will be $2(P \cdot 2\pi a)$, since the distance moved by P in the direction of its line of action is $2\pi a$.

If the average total resistance to the plate at the end of the screw be R, the work done by the screw in one revolution will be $Rp$.

$\therefore$ the efficiency of the machine is $\frac{Rp}{4\pi aP}$.

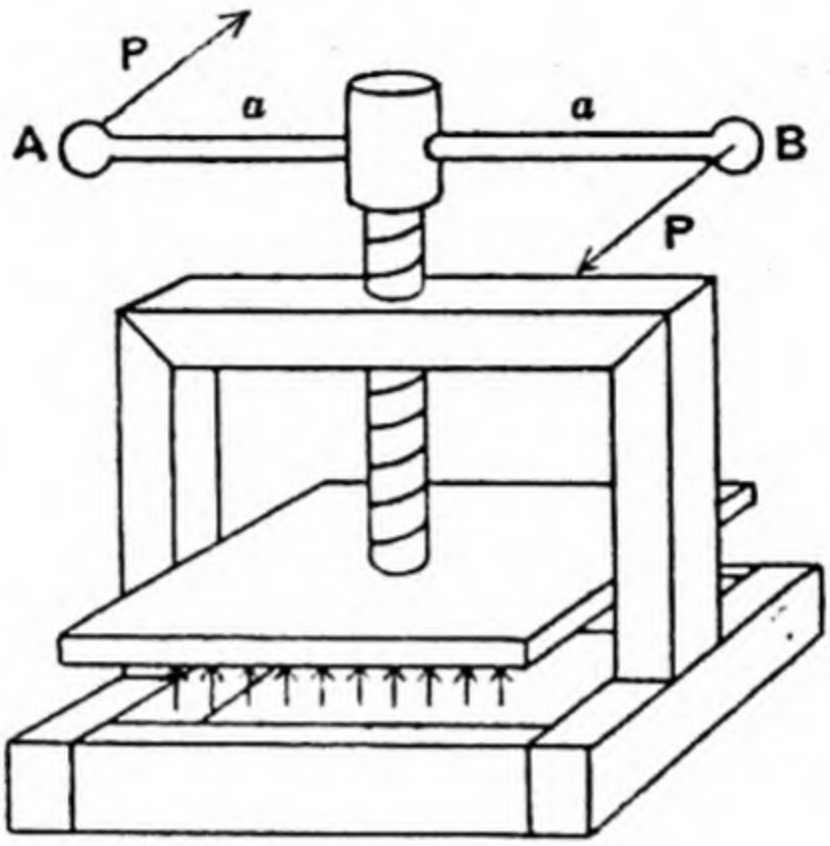



FIG. 59.

## EXAMPLES IX.

**1.** The pitch of a screw-jack is $\frac{1}{2}$ in. and the effort is applied tangentially to an arm 2 ft. long, measured from the axis of the screw. Find the efficiency of the jack if it requires an effort of 15 lbs. to lift a load of $\frac{1}{2}$ ton.

**2.** If the pitch of a screw-jack is $\frac{1}{2}$ in. and the lever arm 20 in. long, find what force is required to raise 2 tons if 60 % of the work expended is lost in friction.

**3.** In a copying press the handle is 8 in. long on each side of the axis. If the pitch of the screw is $\frac{1}{4}$ in., find the pressure caused by a force of 2 lbs. applied tangentially to each end of the handle, supposing friction neglected. What would be the pressure produced if the efficiency is 60 % ?

**4.** A rack and pinion used for raising a sluice gate consist of a toothed wheel working in contact with the teeth of a

vertical bar, called the rack. The wheel, whose diameter is 6 in., is rotated by a force applied tangentially to an arm 2 ft. long. If the weight of the gate and rack is 2 cwt., find the force necessary to raise the gate if 20 % of it is used in overcoming friction.

**5.** A winch is turned by two handles at the end of arms 16 in. long. The small cog has 10 teeth and the larger has 50 teeth. The diameter of the barrel is 8 in. If the efficiency of the machine is 40 %, find the force required to raise 1 ton.

**6.** A screw press has plates 10 in. square and a screw having six threads to the inch. A force of 4 lbs. wt. is applied at each end of a handle whose total length is 30 in.; find the pressure per square inch between the plates, assuming 40 % of the energy is lost.

**7.** The following results were obtained in an experiment with a screw-jack:

| | | | | | | |
|---|---|---|---|---|---|---|
| P Effort, | 12·0 | 13·8 | 15·7 | 17·6 | 19·6 | 21·5 lbs. |
| W Load, | 100 | 120 | 140 | 160 | 180 | 200 lbs. |

Find, from a diagram, the values of $a$ and $b$ which satisfy the equation $W = a + bP$. (Army.)

**8.** A screw moves a slide at the rate of 25 ft. per minute; if the screw makes 200 revolutions per minute, find what its pitch must be.

**9.** Prove that, if a man who can apply a force of 28 lbs. weight uses a lever 3 ft. long to tighten screw bolts 1 in. in diameter, he will be likely to break the bolts if the number of threads to the inch is greater than 8. The breaking tension of iron may be taken as 30 tons wt. to the square inch.

**10.** In the ordinary screw coupling of a railway carriage the pitch of each screw on either side of the lever arm is $\frac{1}{2}$ in., and the length of the lever arm is 1 ft. If a force of 30 lbs, wt. is applied at the end of the arm, find with what force the carriages are pulled together, neglecting friction.

**11.** A bicyclist exerts a downward pressure of 60 lbs. wt. on the pedal, at right angles to the crank which is 7 in. long.

If the diameter of the chain wheel is 10 in., find the pull on the chain.

**12.** During one revolution of the pedals of a bicycle the machine moves forward the same distance as a wheel in one revolution whose diameter is $g$ inches ; $g$ is called the gear of the bicycle. If the length of the crank is 7 in., find the velocity ratio of a bicycle whose gear is 70.

**13.** The pedal wheel of a bicycle has 56 cogs. This wheel is connected by a chain to a wheel with 18 cogs, which rotates with a driving wheel 26 in. in diameter. Find the velocity ratio and the gear of the bicycle, if the crank is 7 in. long.

**14.** If a couple whose moment is C lbs.-ft. rotates a body through an angle $\theta$ radians, prove that the work done by the couple is $C\theta$ ft.-lbs.

## Laboratory Experiments.

**1.** Carry out the experiments on machines as described in the text, pp. 44–64.

**2.** By means of a fixed pulley, cord and pulley shackle, construct a lifting appliance in which the pulley shackle is supported by two parts of the cord which are nearly parallel.

Sketch the arrangement.

Suspend a weight W from the hook of the pulley shackle. Attach a spring balance to the free end of the cord, and note its reading (S) as its ring or hook is pulled steadily by the hand to raise W slowly.

Repeat with eight other values of W, and finally replace W by an unknown load, observing the corresponding value of S

Tabulate your observations.

Give a graph coordinating W and S, and deduce an equation connecting them.

Read from the graph the weight of the unknown load, and also deduce it from the equation. **(Army.)**

# CHAPTER III.

## TRIANGLE OF FORCES. PARALLELOGRAM OF FORCES.

### Graphical Representation of a Force.

In order to specify a force completely, we must know

(1) Its magnitude. (2) Its direction.

If we are considering the rigid body on which it acts, we must also know

(3) Its point of application.

These requirements are satisfied if we represent a force graphically by a straight line. The line can be drawn so that its length represents the magnitude of the force

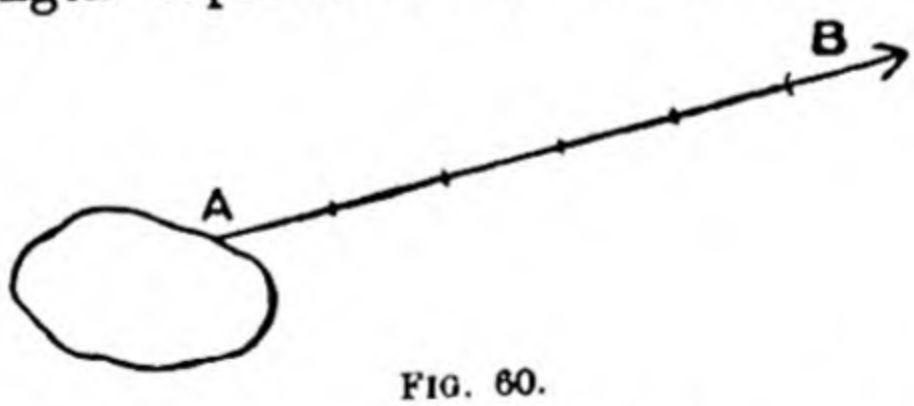



FIG. 60.

according to some scale. Thus 40 lbs. wt. can be represented by a line 4 cms. long on the scale 1 cm. ≡ 10 lbs. wt. The direction of the line represents the line of action of the force ; this is shown by an arrowhead or by the lettering, AB denoting the direction A to B, and BA the opposite direction.

One end of the line represents the point of application of the force.

It has been shown experimentally that the equilibrant of a system of parallel forces, acting vertically downwards, is a force equal to their sum, acting vertically upwards.

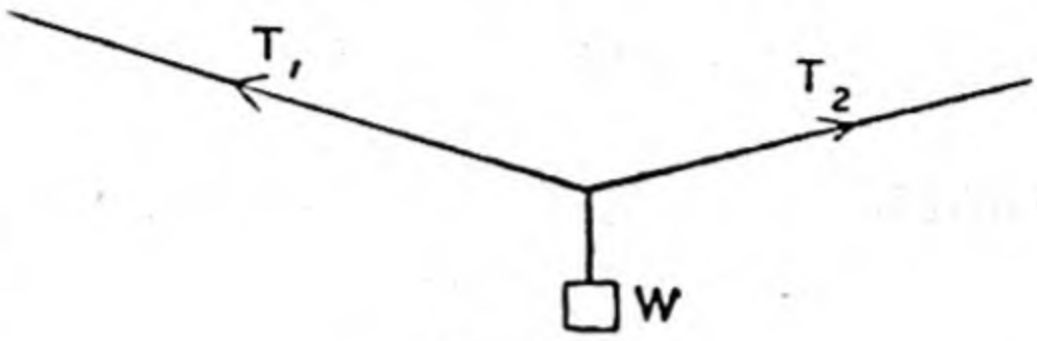



FIG. 61.

If a weight W is supported by two strings, fastened to spring balances, pulling in opposite ways, so that they are nearly horizontal, it will be seen that the pull on either balance is much greater than the weight to be raised.

We require now to find a method of determining the equilibrant of two forces, acting at a point, when the forces make a given angle with one another.

**Vector Addition.**

If from A, I move 5 miles due East to B, my displacement can be completely represented by a straight line AB. It can be drawn to some scale to represent the distance;

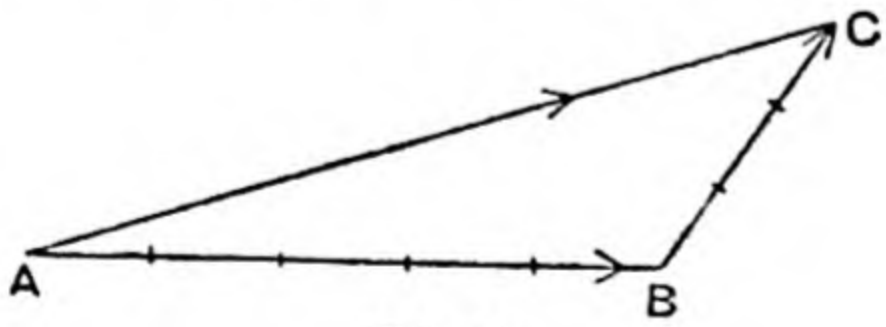



FIG. 62.

the direction of motion can be represented by the direction of the line drawn with an arrowhead; and the final and

initial positions are represented by the extremities of the line. If I now go 3 miles N.E. to C, my new displacement is represented by BC.

The final result of my journey is the same as if I had moved directly from A to C along the line AC, whose length will indicate the distance in this direction, which is equivalent to the other two displacements.

In order to specify my displacement from A to B and from B to C, it is necessary to state two facts: (1) the magnitude of the displacement, and (2) the direction. This information would also be required in specifying a velocity, or a force, whereas an area or a temperature would require merely a number. Quantities which involve both magnitude and direction are called *vector* quantities.

In the case of the displacements illustrated above, we represent the fact that the displacement AC is equivalent to the two displacements AB and BC by the notation

$$\vec{AB} + \vec{BC} = \vec{AC}.$$

The arrow is drawn over the displacements to show that they are vectors, and the operation represented by the statement $\vec{AB} + \vec{BC} = \vec{AC}$ is called vector addition.

The following experiment shows that the resultant of two forces acting at a point, which are not parallel to one another, is given by a similar construction.

*Experiment* 1. Clamp two dynamometers to a board as in the diagram, or support them in retort stands, and fasten to them two strings knotted at A, where a third string supports a weight W.

On the board draw a line from A in the direction of the force $T_1$. Mark off a distance AB representing the magnitude of the force $T_1$, as shown by the dynamometer, on some

convenient scale, say 1 cm. ≡ 10 grams wt. Through B draw BC parallel to the direction of the force $T_2$, and of such length as represents the force $T_2$ on the same scale as before.

If CA be joined, we shall find that its length represents the magnitude of the force W, and that its direction is

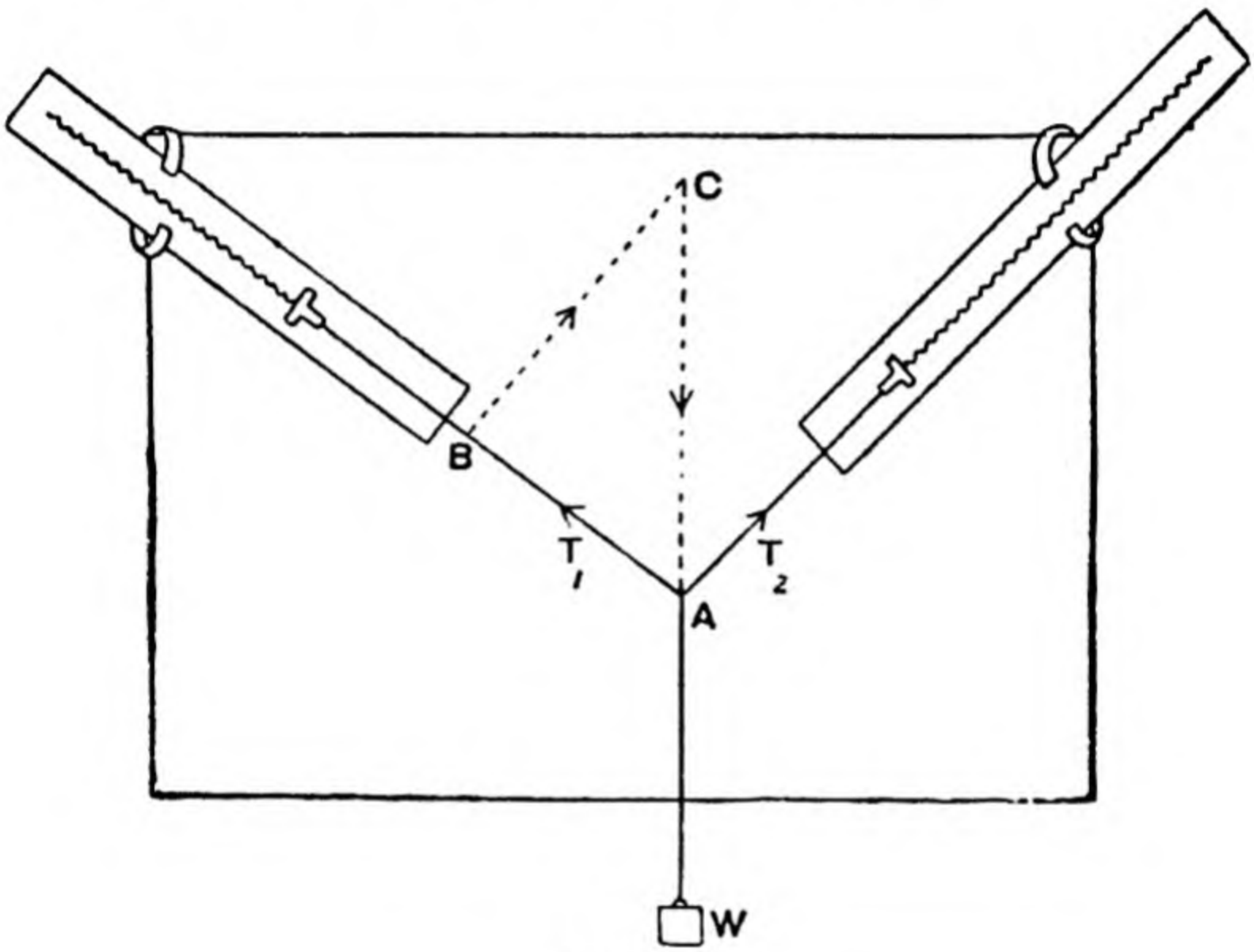



FIG. 63.

vertical, so that it represents the equilibrant of $T_1$ and $T_2$ in magnitude and direction.

Any triangle whose sides are parallel to the sides of ABC will represent equally well the direction of the three forces, and also their magnitude, though on a different scale.

The theorem known as the Triangle of Forces is thus enunciated :

**If three forces acting at a point are in equilibrium, they may be represented in magnitude and direction (but not in point of application) by the sides of a triangle taken in order if these sides are drawn parallel to the three given forces.**

By 'taken in order' is meant that the sides of the triangle must be traversed in the order AB, BC, CA, or AC, CB, BA. It is obvious that the forces represented by AB, BC, AC would not be in equilibrium, if they acted at a point O.

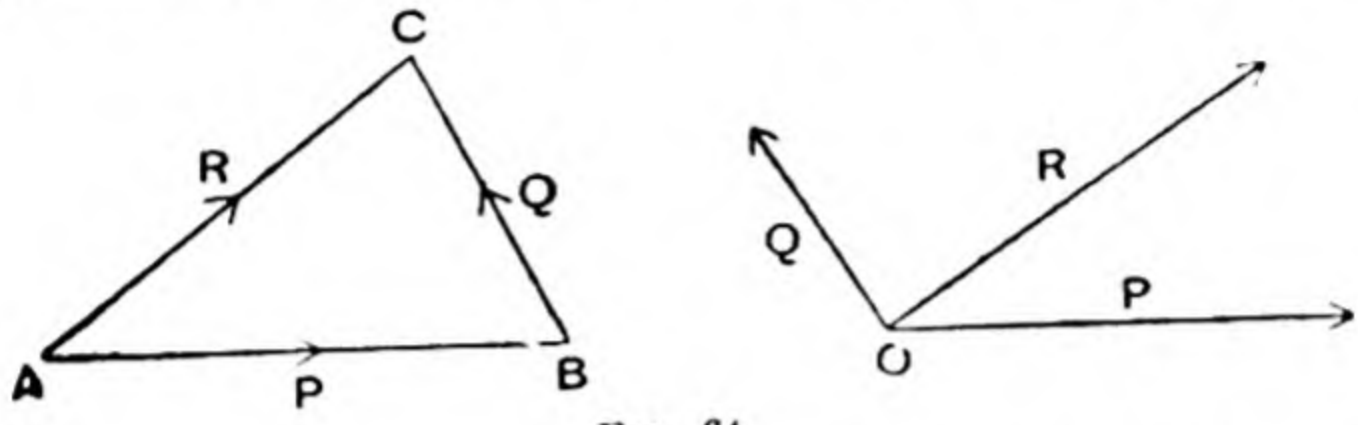



FIG. 64.

The experiment we have performed has shown approximately that this result is true in one case—the general proof of the theorem is best deduced from dynamical principles, and must be postponed.

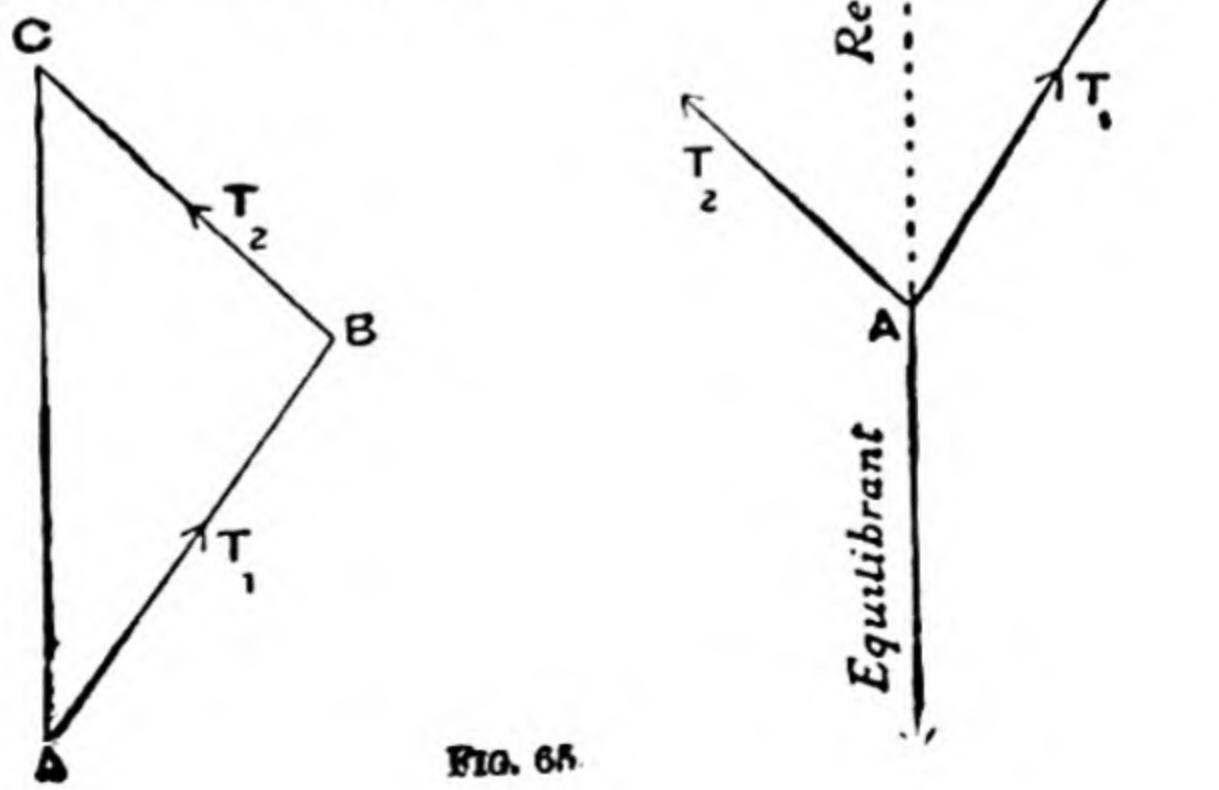



FIG. 65.

Since the force represented by CA is the equilibrant of the forces $T_1$ and $T_2$, if they act at a point, it is clear that if we substituted one single force for the two forces $T_1$ and $T_2$, this single force would be equal to the equilibrant of $T_1$ and $T_2$, but it would act in the opposite direction.

Therefore the *resultant* of $T_1$ and $T_2$, if they act at a point, is represented, in magnitude and direction, by AC; that is to say,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

*Experiment* 2. A string is fastened to a spring balance B and to the end of a light rod CA; a weight W of 1 Kg.

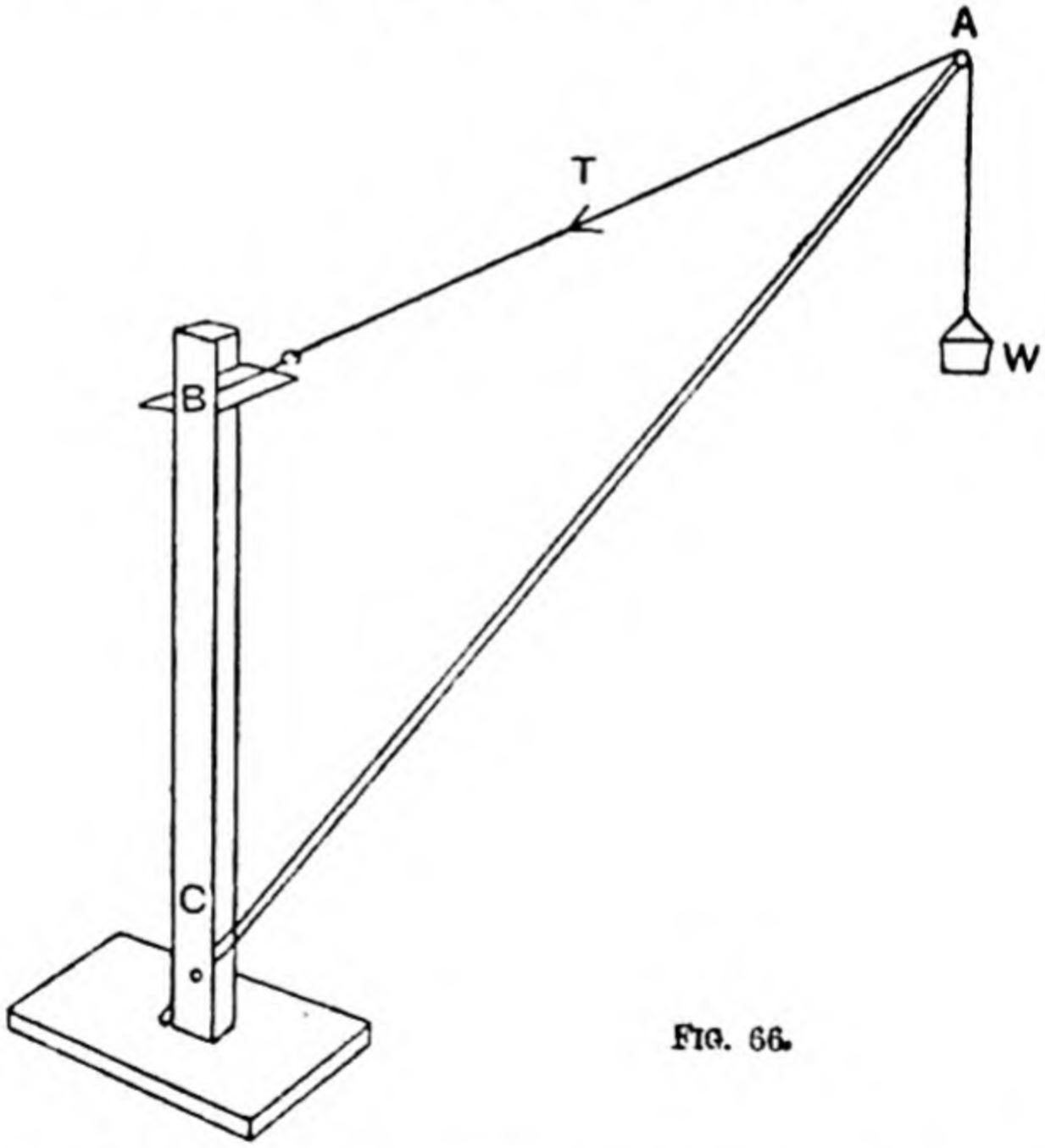



FIG. 66.

hangs from A. This apparatus represents a crane CA is called a strut and AB is called the tie.

To find the direction of the thrust at A exerted by the rod CA.

Consider the equilibrium of the rod CA.

If W is considerable, the weight of the rod may be neglected, and it is then in equilibrium under a force (P) at C and the resultant (R) of T and W at A, that is to say under two forces only, P and R, which, therefore, act along AC in opposite directions. Therefore the thrust *of the rod* at A must equal R, and will be in the direction CA.

Now, consider the forces at A; they are:

(1) A tension T along AB. (2) The weight W.

(3) The thrust of the rod CA along CA.

Hence a triangle of forces may be drawn.

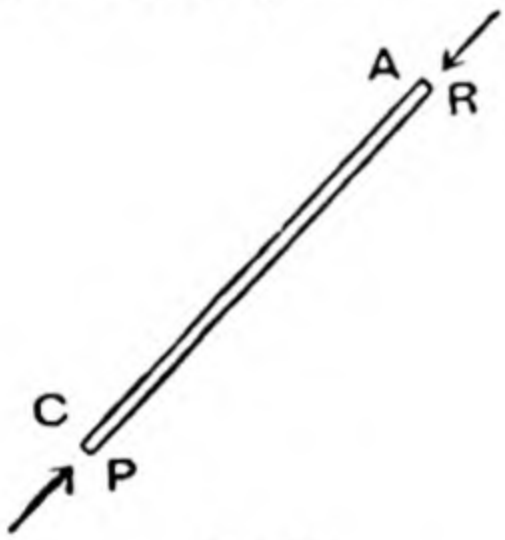



FIG. 67.

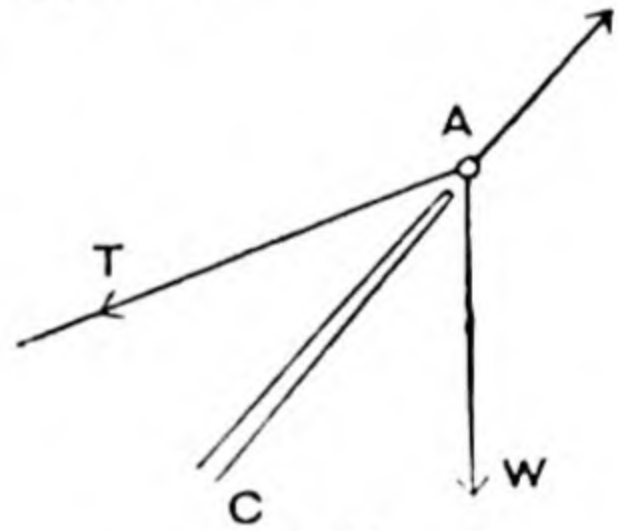



FIG. 68.

The triangle ABC has its sides parallel to these three forces. Therefore, if BC represents W, AB on the same scale represents T. Calculate this force, and compare with the reading of the spring balance. If the rod is fastened to a spring balance at C, placed so that the pressure of the rod extends the spring, the thrust of the rod may be measured and compared with the calculated value.

*N.B.*—We state here that the thrust of the rod CA acts at the point A. Of course, strictly speaking, forces do not act at a point, but it may perhaps make the action of the forces at A clearer if we suppose them all to act on a small particle at that point. Under the action of

T and W this particle would move. It is kept at rest by the pressure of the rod CA, so that the resultant of T and W must act along the line AC.

EXAMPLE 1. A weight of 50 lbs. is supported by two strings AO 5 ft. and OB 8 ft., fastened to two points A and B in the same horizontal line, 10 ft. apart. To find the tensions in the strings.

Draw the diagram to scale 1 ft. ≡ 1 cm. The point O is in equilibrium under the action of three forces. $T_1$ along OB, $T_2$ along OA and 50 lbs. vertically downwards. Therefore we can draw the triangle of forces OMN. Scale 1 cm. ≡ 5 lbs.

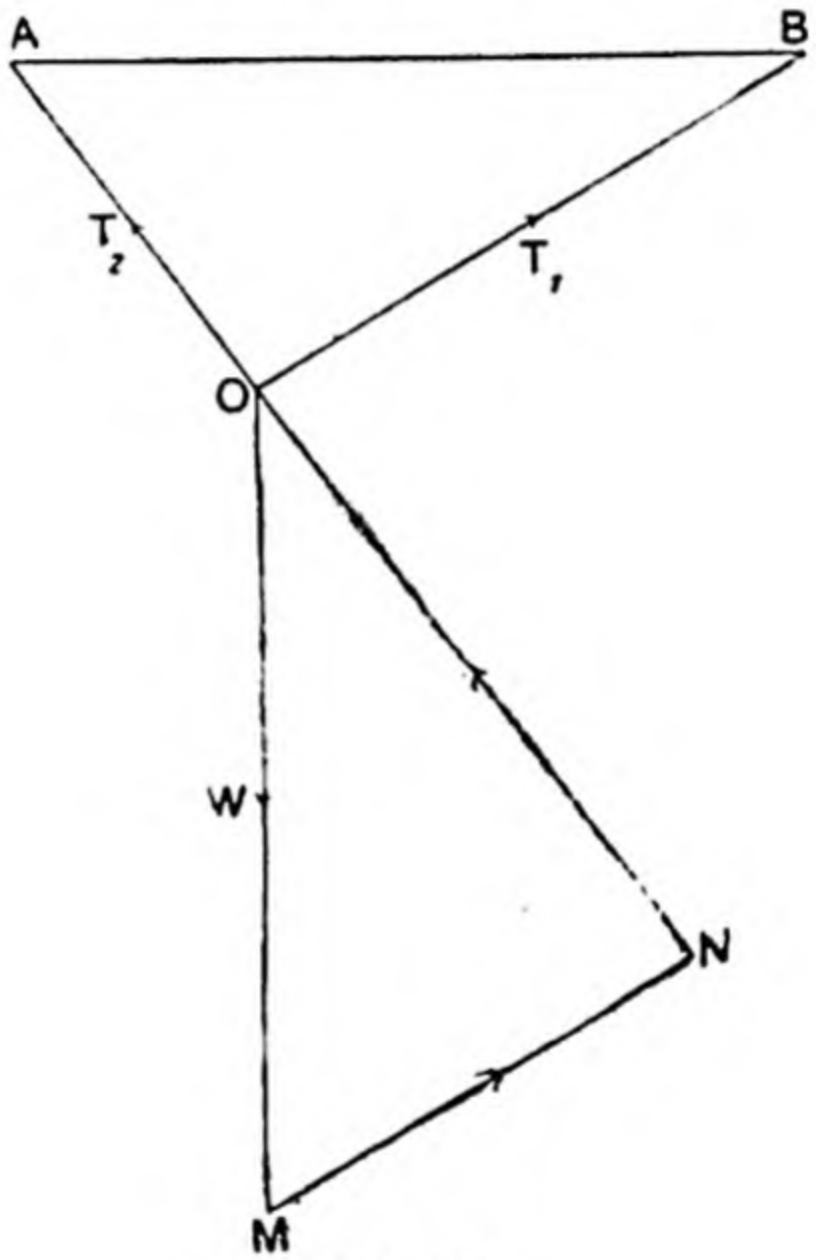



FIG. 69.

Mark off OM = 10 cms. to represent 50 lbs.
Produce AO to meet MN drawn parallel to OB.

MN = 6·2 cms ; ∴ $T_1$ = 31 lbs.
NO = 8·7 cms. ; ∴ $T_2$ = 43·5 lbs.

**Example 2.** A weight is attached to a smooth ring, which slides on a string whose ends are fastened to two pegs in a wall not at the same height. If the string is longer than the distance between the pegs, prove that the two parts of it will be equally inclined to the vertical.

Since the ring is smooth, the tension in the string will be the same throughout.

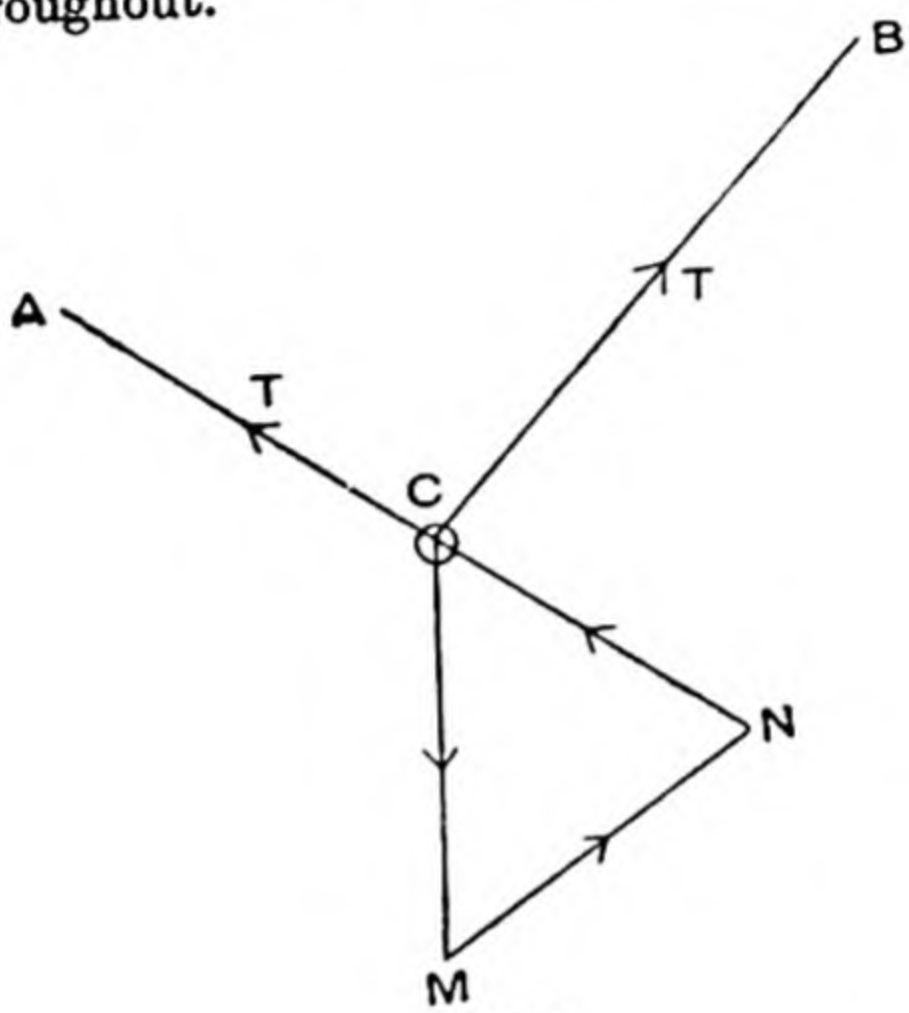



Fig. 70.

The point C is in equilibrium under the action of three forces; T along CA, T along CB and W the weight of the ring.

If we draw CM to represent the weight of the ring and complete the triangle of forces, it is clear that this triangle CMN must be isosceles, since MN and NC represent equal forces T.

Therefore MN, CN make equal angles with the vertical CM.

Therefore CB, CA, which give the direction of the forces represented by MN and NC, also make equal angles with the vertical.

## EXAMPLES X.

**1.** A weight of 50 lbs. is suspended by two ropes 12 ft. and 15 ft. long, fastened to two points on the same horizontal line, 18 ft. apart. Find the tensions in the two ropes.

**2.** A weight W is supported by two strings AB, AC in a vertical plane. AB is fastened to a nail at B and makes 40° with the vertical; AC passes over a pulley at C and supports a weight of 15 lbs. If AC makes 50° with the vertical, find W.

**3.** A weight of 50 lbs. hangs by a long rope from a point A. A string is fastened to this rope at B, 12 ft. from A, and is pulled horizontally by a force P, until B is 5 ft. from the vertical through A. Find the force P required to keep B in this position.

**4.** A weight of 20 lbs. is supported in a vertical plane by two forces of 12 lbs. and 15 lbs. Find the angles these forces must make with the vertical.

**5.** A rope 5 ft. long, which will not lift a weight of more than 50 lbs., is tied to two points on the same horizontal line 3 ft. apart. Find the greatest weight that can be fastened to the middle of the rope in this position without breaking it.

**6.** A rope AB, 5 ft. long, is fastened to a point A and carries a weight of 20 lbs. at B. The weight is pulled to one side by a force P at B, so that AB makes 30° with the vertical. Find graphically, on one diagram, the value of P when its direction makes angles of 50°, 70°, 90°, 110°, 130° with AB.
For what direction is P a minimum?

**7.** The vertical pole of a jib crane is 20 ft. long, the strut is 30 ft. and the tie 25 ft. Find the forces exerted by the strut and the tie if the weight supported is 1 ton. Neglect the weight of the strut.

**8.** A force of 20 lbs. acts at 20° to the horizontal. Find what horizontal and vertical forces would be required to balance it.

**9.** Two rafters AB, BC, each 20 ft. long, are fastened to the tops of two walls at A and C, on the same level, 36 ft. apart. If a vertical force of 1 cwt. acts downwards at B, find the additional thrust in the rafters and the horizontal pressures exerted at A and C.

**10.** A load of 3 cwt. is suspended from a point A by a rope. If a force is applied to the rope 10 ft. from A, find the least value of this force which will keep the load 4 ft. from the vertical through A.

**11.** The strut of a jib crane is 18 ft. long and the tie, 14 ft., is fastened to the upright, 12 ft. above the bottom of the strut. If the strut cannot support a greater thrust than 1 cwt., find what load on the crane will make the strut break.

**12.** One end of a long string is tied to a horizontal beam at A, and the other end is passed over a small pulley on the beam at B, 6 ft. from A. If a weight of 600 grams is fastened to the string at C, 2 ft. from A, find what weight must be fastened to the the end of the string over the pulley if AC makes 40 ° with the horizontal.

**13.** A light horizontal rod AC, 3 ft. long, is hinged at A, and is supported at C by another light rod BC, 5 ft. long, which is hinged at B, a point vertically below A. If a weight of 50 lbs. is suspended from C, find the thrust in the rod BC and the tension in AC.

**14.** A and B are two points, 4 ft. apart, A being 1 ft. higher than B. A string 6 ft. long is fastened to A and B. A smooth ring runs on the string and carries a weight of 100 lbs. Find, by drawing, the inclination of each part of the string to the vertical and the tension in the string.

**15.** A triangular framework is made of light rods, whose lengths are AB 12 ft., BC 8 ft., CA 10 ft. If the framework is placed in a vertical plane with AB on a smooth horizontal surface, find what tension will be produced in AB if a vertical pressure of 1 cwt. is applied at C.

**16.** ABCD is a rope fastened to a horizontal beam at A and D. When a weight of 12 lbs. is hung at B, and a weight of W lbs. at C, the part AB of the rope makes 60° with AD, BC is 25° below the horizontal through B, and CD is inclined at 40° to the vertical. Find W.

*N.B.*—Draw the triangles of forces for the points B and C, so that the same line represents the tension in BC.

**17.** If AB (cf. Qu. 16) is inclined at 50° to the vertical when there is a weight of 4 cwt. at B and 6 cwt. at C, find the direction of CD and the tension in it, given that the tension in AB is 6 cwt.

**Parallelogram of Forces.**

If in Fig. 71 we complete the parallelogram ABCD, then, since AD is equal and parallel to BC, it represents the magnitude and direction of $T_2$, so that AC, which represents the resultant of $T_1$ and $T_2$, is the diagonal of the parallelogram, of which the lines representing $T_1$ and $T_2$ form adjacent sides.

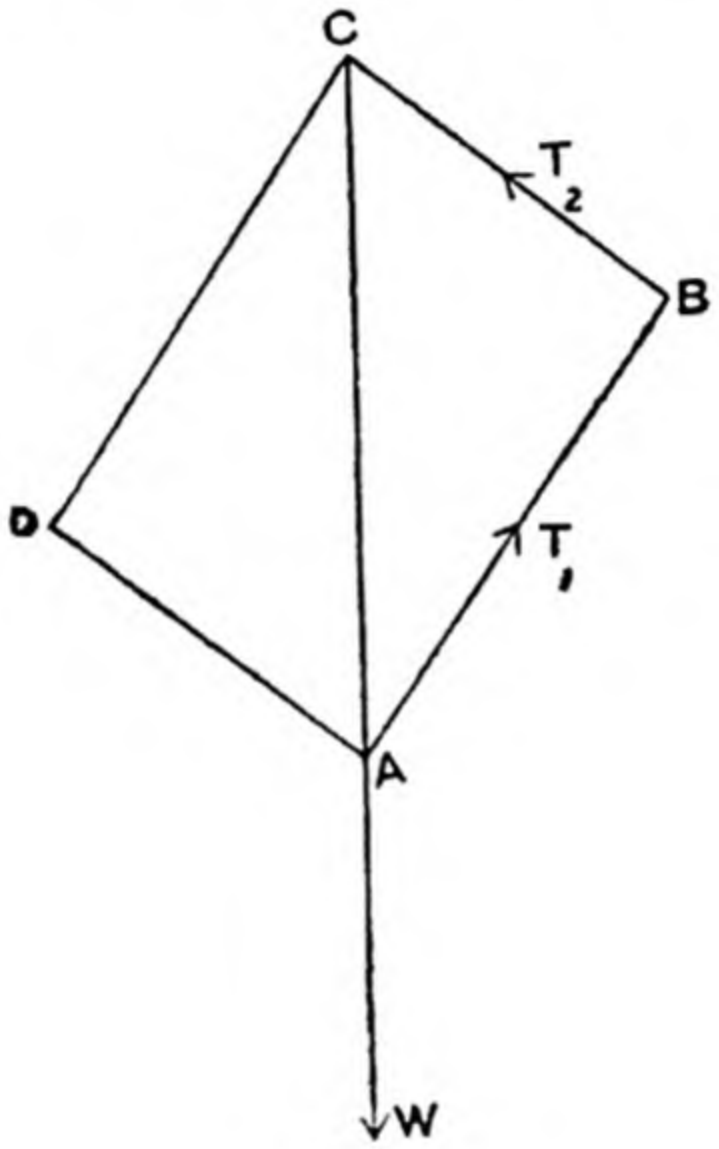



FIG. 71.

This is the original form of the theorem which is known as the *Parallelogram of Forces*. It may be enunciated thus:

If two forces, acting at a point, be represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from the point, their resultant is represented on the same scale, both in magnitude and direction, by the diagonal of the parallelogram drawn from the point.

If the two forces $T_1$ and $T_2$ are nearly opposite, a slight error in their direction would make a considerable error in the magnitude of the resultant. In such a case the value of R is more accurately obtained by calculation.

**Results by Calculation.**

The resultant of two forces P and Q, acting at a point and inclined to one another at an angle $\alpha$, may be found by calculation.

If the sides OA, OB of the parallelogram OACB represent in magnitude and direction the forces P and Q acting at O, then the diagonal OC represents their resultant R.

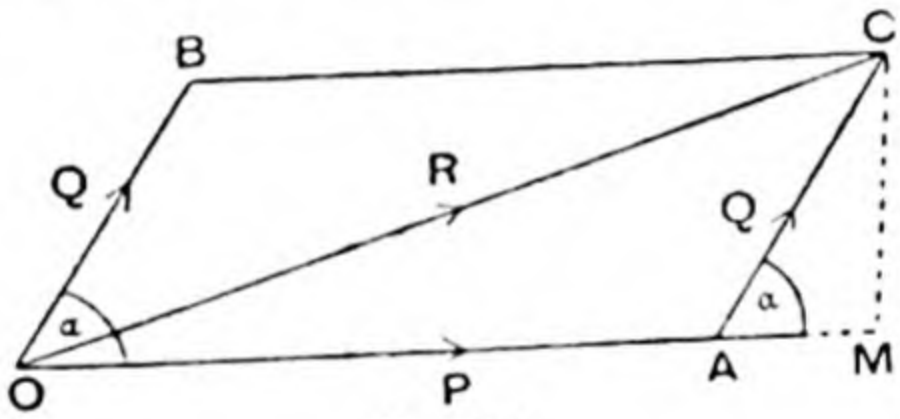



FIG. 72.

Draw CM perpendicular to OA.

Then $CM = AC \sin \alpha$ and $AM = AC \cos \alpha$;

$$\therefore OC^2 = (OA + AC \cos \alpha)^2 + (AC \sin \alpha)^2$$

$$= OA^2 + AC^2 + 2OA \,.\, AC \,.\, \cos \alpha,$$

*i.e.* $$R^2 = P^2 + Q^2 + 2PQ \cos \alpha,$$

since the forces P, Q, R are proportional respectively to the lengths OA, OB(AC), OC.

Otherwise, by trigonometry,

$$OC^2 = OA^2 + AC^2 - 2OA \,.\, AC \cos \hat{OAC}$$

$$= OA^2 + AC^2 - 2OA \,.\, AC \cos (180 - \alpha)$$

$$= OA^2 + AC^2 + 2OA \,.\, AC \cos \alpha.$$

The angle $\theta$, between P and the resultant R is given by

$$\tan\theta = \frac{MC}{OM} = \frac{AC\sin\alpha}{OA + AC\cos\alpha} = \frac{Q\sin\alpha}{P + Q\cos\alpha}.$$

If P and Q are at right angles, we have

$$R^2 = P^2 + Q^2.$$

## EXAMPLES XI.

1. Find the resultant of two forces of 5 lbs. wt. and 10 lbs. wt., acting at a point at right angles to one another.

2. Find the resultant of two forces of 12 lbs. wt. and 8 lbs. wt. inclined to one another (i) at 60° ; (ii) at 120°.

3. Two forces inclined to one another at an angle of 50° have a resultant of 8 lbs. wt. If one force is 5 lbs. wt., find the other.

4. Two equal forces inclined to one another at 60°, act at a point and are kept in equilibrium by a third force of 8 lbs. wt. Find the two equal forces.

5. The pull in each part of the string which passes over a nail and supports a picture is 1 lb. wt. If the two parts of the string are inclined to one another at 45°, find the resultant pull on the nail. What is the weight of the picture ?

6. ABCD is a parallelogram. Forces act at B along AB and BC, proportional to the lengths of these lines. Which diagonal of the parallelogram represents the magnitude of the resultant ? Explain your answer.

7. ABCD is a square of side 1 ft. It lies on a table, and can turn in a horizontal plane round a pin at A. If it is pushed with a force of 1 lb. wt. along BA, and pulled with a force of 2 lbs. wt. along AD, find the magnitude and direction of the resultant force on the pin.

8. Two unequal forces inclined to one another at 70° have a resultant of 10 lbs. wt., which makes an angle of 30° with one of the forces Find the two forces.

9. Two equal light rods AB, AC are hinged at A, and placed in a vertical plane, with B and C hinged to two points in a horizontal line. If the rods are inclined to one another at

175°, find what thrust will be produced along each of them if a vertical force of 10 lbs. wt. is applied at A.

**10.** The breaking tension of a brass wire is $3{\cdot}4 \times 10^6$ grams per sq. cm. One end of the wire is securely fastened, a smooth ring carrying 5 Kg is threaded on the wire, and the other end is then pulled. Find the angle between the two parts of the wire when it breaks, if the diameter of the wire is 1 mm.

**Resolution of Forces.**

The figure represents a railway wagon being pulled along the line by a horse at the side of the line. The rope is inclined at an angle $\alpha$ to the direction in which the wagon moves, and P represents the pull of the rope. If the angle $\alpha$

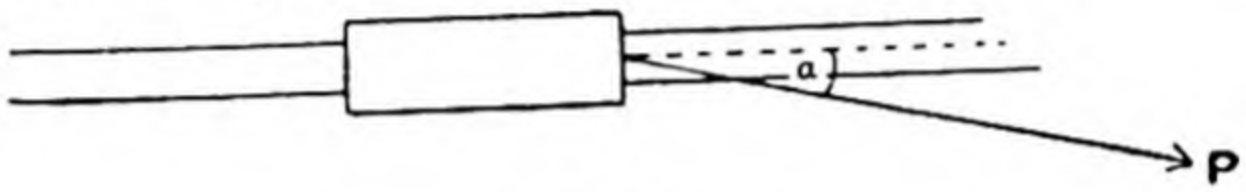



FIG. 73.

were 90°, the pull of the rope would not move the wagon forward at all, but would tend to pull it over sideways, and it is clear that the horse would pull with most effect if he walked between the rails.

Suppose $\alpha$ to be 30° and P to be 50 lbs.: we require to find how much of the force P is effective in moving the wagon.

Draw AB representing the magnitude and direction of P,

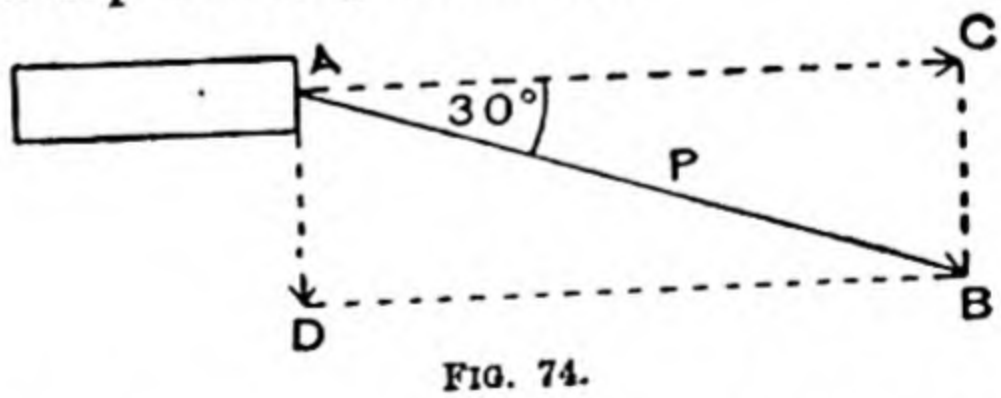



FIG. 74.

and draw BC perpendicular to the direction in which the wagon moves. Complete the rectangle ACBD.

By the parallelogram of forces we know that AB represents the resultant of two forces, which are represented in magnitude and direction by AC and AD, so that we can let these two forces take the place of P. The force AD pulls the wagon sideways against the rails, while the force AC drags the wagon forward. If AB represents 50 lbs., we see that AC represents $50 \cos 30°$, and AD $50 \sin 30°$.

The effective pull of the horse when $\alpha$ is 30° is therefore $50 \cos 30° = 25\sqrt{3}$ lbs.

## EXAMPLES XII.

**1.** A heavy ring hanging on a horizontal rod is pulled along the rod by a string inclined at 20° to the rod. If the tension in the string is 4 lbs., find how much of this force is effective in moving the ring.

**2.** A boy pulls a sledge along the road with uniform velocity by means of a rope over his shoulder. If the pull in the rope is 30 lbs. when it is inclined at 60° to the ground, find the force that is effective in moving the sledge forward.

**3.** A barge is being towed along a river by a rope 30 ft. long. If the boat is 10 ft. from the path, and the rope is pulled with a force of 30 lbs. wt., find the force which resists the motion of the boat through the water.

**4.** If the resistance to the motion of the boat in Qu. 3 were 50 lbs., find with what force the rope would have to be pulled.

**5.** If the length of the tow rope were increased to 50 ft., find what force would then be required to move the barge in Qu. 3. Explain why it is easier to tow a boat with a long rope than with a short one.

**6.** Two strings are fastened to a ring on a smooth rod, so that it is pulled in opposite directions. If one string, which is inclined to the rod at 20°, is pulled with a force of 10 lbs. wt., with what force must the other be pulled to keep the ring at rest if this string is inclined to the rod at 40° ?

**7.** A rope is fastened to a vertical pole, 6 ft. from the ground. and the pole is broken when the rope is pulled at an angle of 50° to the vertical with a force of 80 lbs. wt. What force would have broken the pole if the rope had been pulled horizontally ?

**8.** A picture is hung up by a string 3 ft. long, passing over a nail, and fastened to two rings on the picture at the same level, 2 ft. apart. If the string lies in a vertical plane and the tension throughout is 1 lb., find what the tension would have been if the string had been cut in half, and each part fastened to a nail so as to be vertical.

**9.** In the first part of Qu. 8, find the pressure the string produces on the nail.

**10.** A body weighing 10 lbs. rests on a rough plane inclined at 20° to the horizontal. Find two forces, one acting down the plane, the other at right angles to it, which together would be equivalent to the weight of the body. Hence, find the magnitude of the frictional force which is preventing the body from sliding down the plane.

The operation of decomposing a single force into two forces at right angles to one another, which together have the same effect as the single force, is called " Resolving the force into two directions at right angles." The two forces are then called the Resolved Parts of the given force in these two directions.

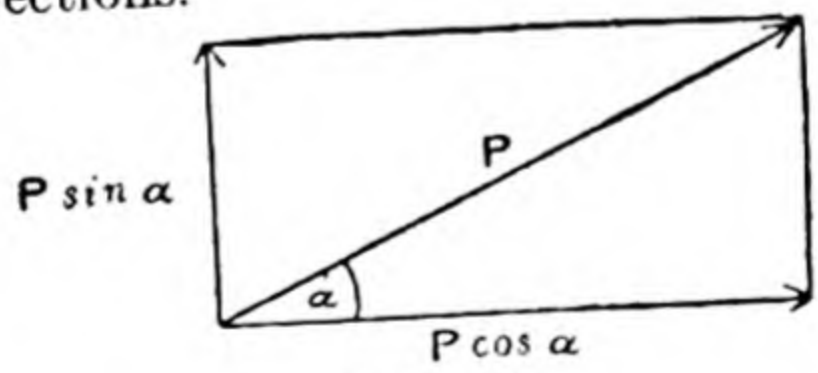



FIG. 75.

If one direction makes an angle $\alpha$ with the force P, the resolved part in this direction is $P \cos \alpha$, and in the other direction $P \sin \alpha$.

If $\alpha$ is obtuse, the resolved part, $P\cos\alpha$, in the direction OA will be negative, showing that this force acts in the direction OA′.

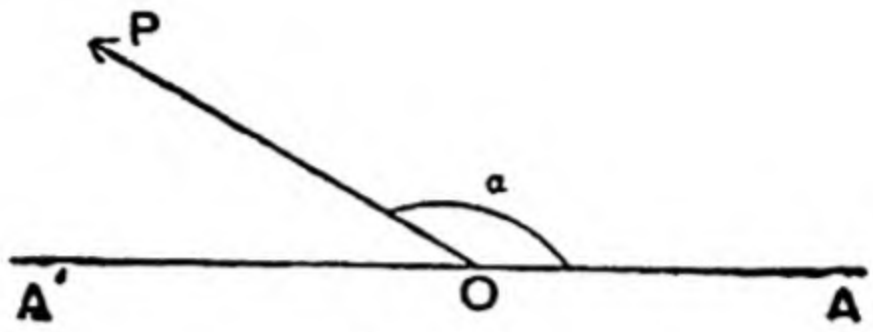



FIG. 76.

It will now be clear that the pressure exerted by a smooth surface on a body in contact with it will be in a direction at right angles to the surface.

If a body of weight W is placed on a smooth inclined plane, the pressure of the plane must be entirely in the direction ON.

Suppose the pressure to be along OR; then this force may be resolved into two components, one along the plane OP

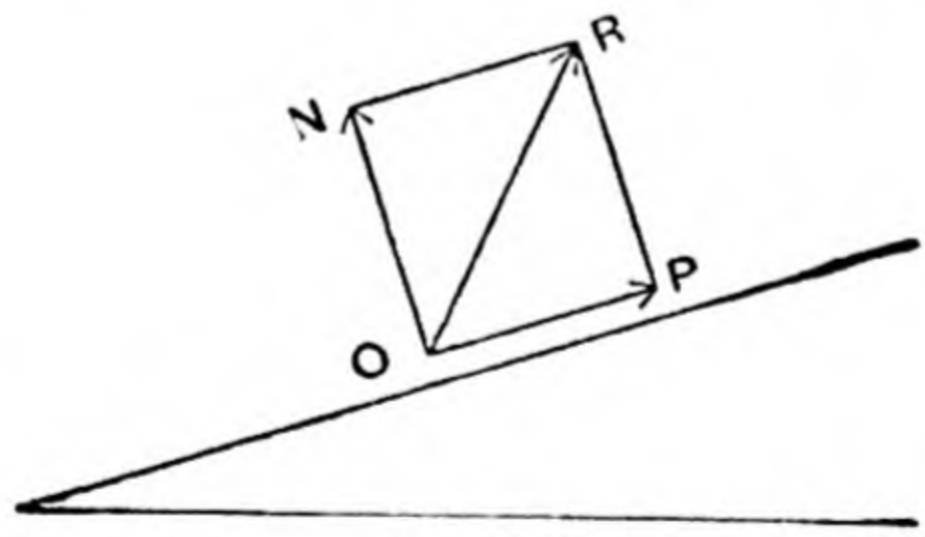



FIG. 77.

and the other at right angles to it along ON. But, since the plane is smooth, it can exert no frictional force such as OP. Therefore the pressure must be entirely at right angles to the surface.

**Work.**

In Chapter II., p. 41, it was pointed out that the work done by a force P, while its point of application moves from A to B, is measured by P(AC), and not by P(AB).

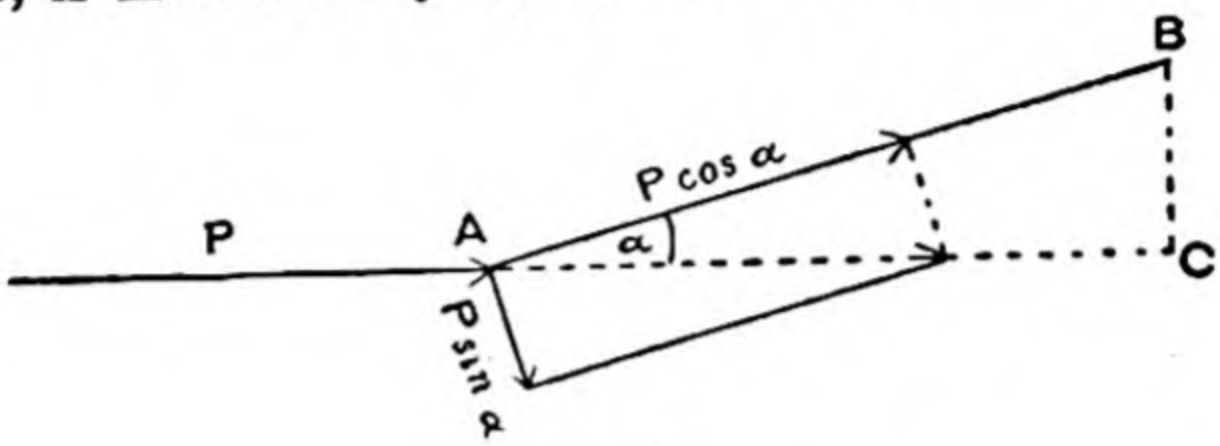



FIG. 78.

Resolve P into two components, one $P\cos\alpha$ along AB and the other $P\sin\alpha$ at right angles to AB.

The work done by the component $P\cos\alpha$ will be $P\cos\alpha(AB)$, while the work done by $P\sin\alpha$ will be zero, since $P\sin\alpha$ is at right angles to AB. Therefore the total work done is $P\cos\alpha(AB) = P(AB\cos\alpha) = P \,.\, AC$.

EXAMPLE. A ring at A, free to move on a rough rod AB, is pulled by a force P in a direction making an angle $\alpha$ with the rod. To find the work done in moving the ring from A to B.

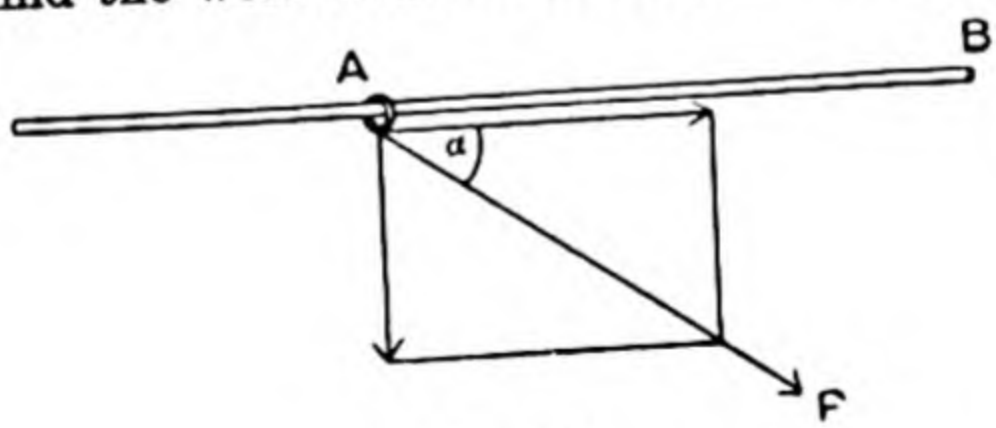



FIG. 79.

Resolve P into two components, $P\cos\alpha$ along the rod and $P\sin\alpha$ at right angles to the rod.

Then the work done by $P\cos\alpha$ is $P\cos\alpha(AB)$, while the work done by $P\sin\alpha$ is zero.

**Sailing into the Wind.**

By resolving the force due to the pressure of the wind we can show how it is possible for a boat to sail in the direction AB when the wind is blowing in the direction CO. Let DOE represent the sail. The pressure on the sail caused by

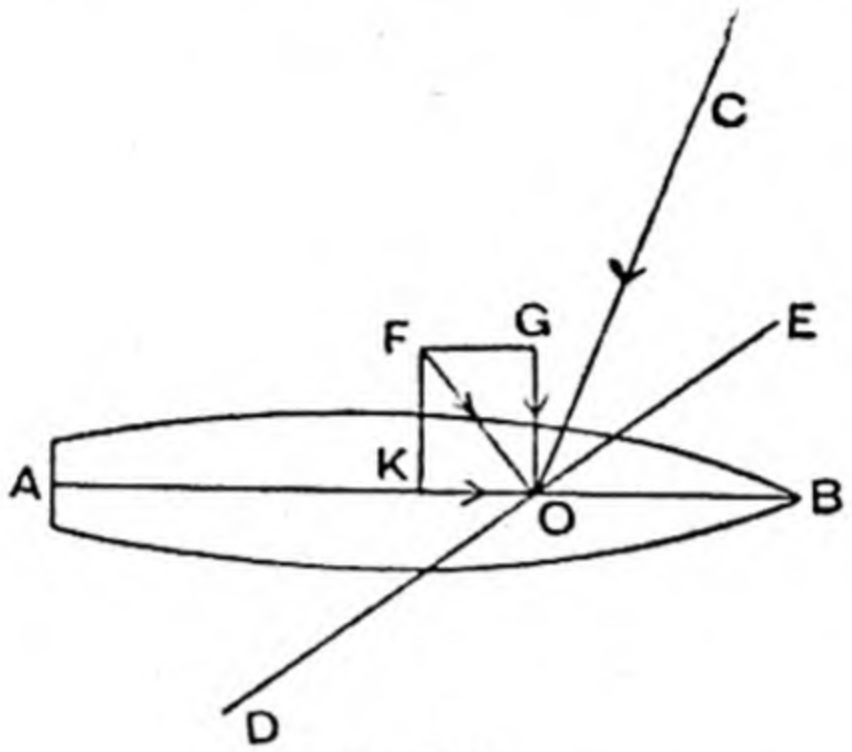



FIG. 80.

the wind will be approximately at right angles to the surface of the sail. Let this be represented by a length FO. Now, resolve the force FO into two components, KO along AB and GO perpendicular to it. The force represented by KO will drive the boat in the direction AB, the other force GO tends to produce leeway, which is to some extent counteracted by the pressure of the water on the deep keel.

EXAMPLE 1. A body of weight W is supported on a smooth plane inclined at $\alpha$ to the horizontal by a force P parallel to the line of greatest slope. Find the magnitude of P

Since the plane is smooth, the pressure N of the plane on the body is entirely at right angles to the surface, *i.e.* normal to the surface.

We may resolve the force W into two components, one parallel to the plane, the other at right angles to it. It will be

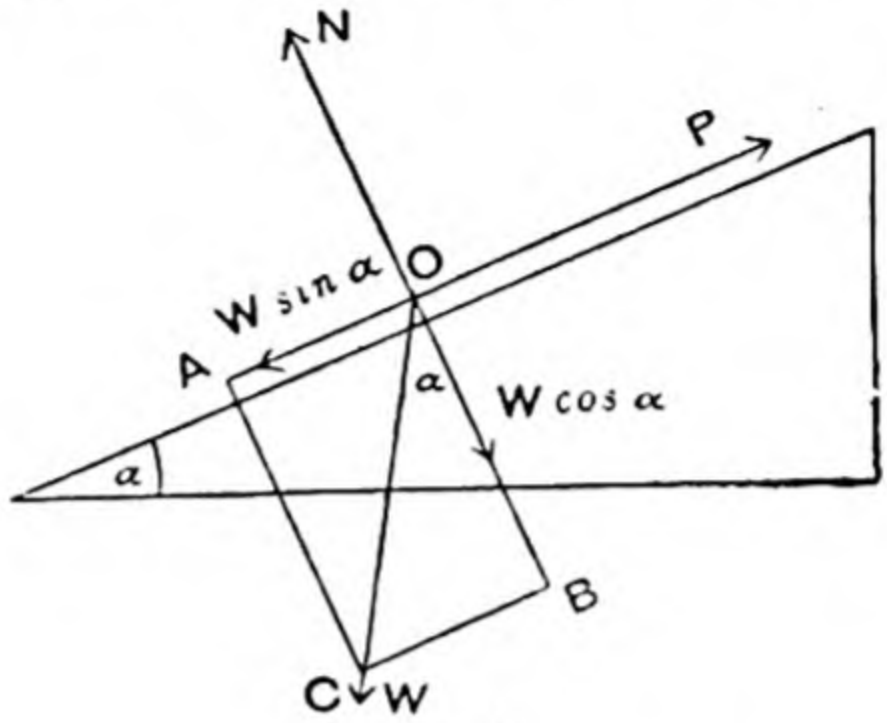



FIG. 81.

clear from the geometry of the figure that the angle COB = α. Therefore these two components will be W sin α and W cos α.

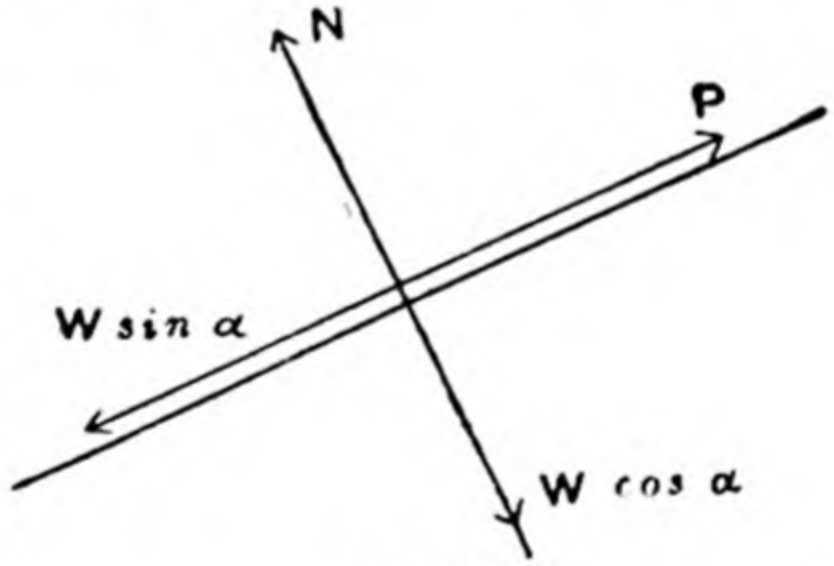



FIG. 82.

The three forces originally acting on the body would have the same effect as the four forces P and W sin α along the plane N and W cos α perpendicular to the plane.

Since the body is in equilibrium, we have

$$P = W \sin \alpha \quad \text{and} \quad N = W \cos \alpha.$$

*N.B.*—The two directions at right angles, which we select for resolving any force, will depend upon the particular circumstances of the problem.

If the force P were horizontal in the above case, we should not resolve W, but we should resolve N horizontally and vertically, since the other forces already act in these directions.

Prove that in this case $P = W \tan \alpha$, $N = W \sec \alpha$.

EXAMPLE 2. A rectangular board ABCD can turn in a vertical plane about A. If it weighs 5 lbs., find what force at C, inclined at 20° to the horizontal, will keep the edge AB horizontal. AB = 3 ft., BC = 2 ft.

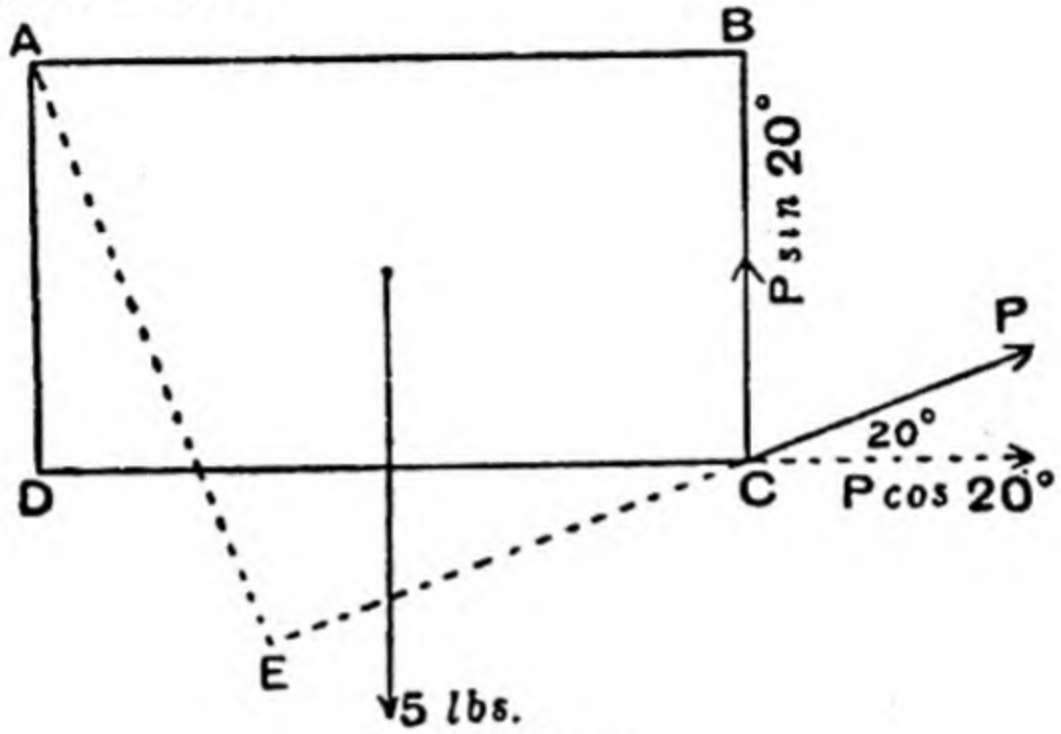



FIG. 82(a).

Take moments about A.

Instead of calculating the length of the perpendicular AE drawn from A to the line of action of P, it will be much simpler to resolve P into two components, $P \cos 20°$ along DC and $P \sin 20°$ along CB, and then take moments about A.

We have $P \cos 20(2) + P \sin 20(3) = 5(\frac{3}{2})$;

$$\therefore P = \frac{15}{4 \cos 20 + 6 \sin 20} = 2{\cdot}6 \text{ lbs.}$$

## EXAMPLES XIII.

1. Two forces each equal to P act at a point, and their resultant also equals P. Find at what angle the forces must act.

2. Two forces, each equal to P, act at an angle $\alpha$ to one another. Prove that the resultant equals $2P \cos \frac{\alpha}{2}$, and that it bisects the angle between the forces.

**3.** A weight of 5 lbs. is fastened to a ring which slides on a smooth rod inclined at 15° to the horizontal. What force parallel to the rod is required to keep the ring stationary ?

**4.** Find what force, acting parallel to a line of greatest slope, will support a weight of 100 lbs. on a plane inclined at 40° to the horizontal.

**5.** A weight of 40 lbs. is slowly drawn up a smooth plane inclined at 20° to the horizontal by a force acting up the plane. What work is done against gravity if the height of the plane is 10 ft ? Calculate the force necessary to keep the weight moving uniformly. Find the distance through which this force acts, and calculate the work done by it.

**6.** A horse pulls a wagon along a railway line by means of a rope inclined at 40° to the line. If the tension in the rope is 30 lbs. wt., find how much work the horse does in moving the wagon 30 ft.

**7.** Find what force acting horizontally would be necessary to prevent a weight of 1 ton from slipping down a smooth plane inclined at 20° to the horizontal.

**8.** A train of 120 tons wt. is ascending an incline of 1 in 100 with uniform velocity. If the friction and resistances are equivalent to 8 lbs. wt. per ton, find the total pull required of the engine.

**9.** Find how many ft.-tons of work the engine in Qu. 8 does per second if the velocity of the train is 20 miles per hour.

**10.** One end of a wire is fastened to the top of a vertical pole 35 ft. high, and the other is fixed to the ground 12 ft. from the bottom of the pole. If the tension in the wire is 30 lbs. wt , find the horizontal pull produced on the pole, and calculate the moment of the tension about the bottom of the pole.

**11** A string stretched to a tension of 10 lbs. wt. passes over a nail, so that the two parts of the string are inclined to one another at 40°. Find the pull on the nail.

**12.** A force of 10 lbs. and a force of P lbs. act at a point. Their resultant is 8 lbs., and it makes an angle of 50° with the 10 lbs. Find the value of P.

**13.** A weight of 5 lbs. is fastened to a string which passes over a smooth pulley and is held at the other end, so that the string on that side of the pulley is horizontal. Find the magnitude and direction of the pressure the string produces on the pulley.

**14.** A block of stone rests on a rough horizontal plane, and is pulled by a force of 10 lbs. wt. acting at 30° to the horizontal, which just fails to move the block. Find the force of friction which is preventing the motion of the block.

**15.** A weight of 5 lbs. hangs by a rope from a point A. A string is fastened to the rope at B, 3 ft. from A, and is pulled horizontally with a force P until the rope makes an angle $\alpha$ with the vertical. If T is now the tension of the rope, show, by resolving this force vertically and horizontally, that

$$T\cos\alpha = 5 \quad \text{and} \quad T\sin\alpha = P.$$

**16.** A rectangular block of stone, weighing 2 tons, is 5 ft. high, and stands on its end, which is 2 ft. square. Find what horizontal force, acting at right angles to the highest edge, will keep the block in equilibrium when it has turned through 20° round its lowest edge.

## Historical Note.

A special case of the parallelogram of forces (when the two forces are at right angles) was discovered by Stevinus (1548-1620), who was led to the theorem by an ingenious investigation as to the condition of equilibrium for a body on an inclined plane.

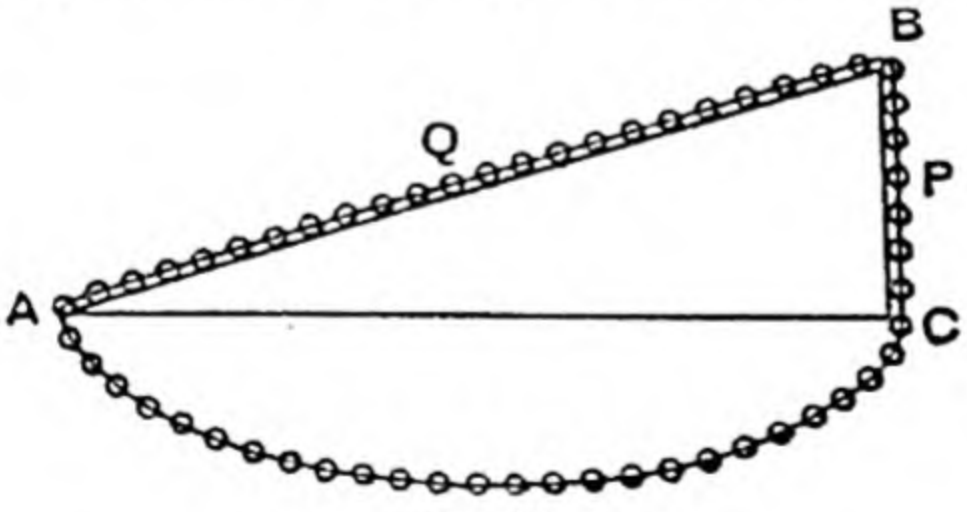



FIG. 83.

His argument was as follows:

An endless uniform chain placed on a smooth inclined plane

would remain at rest. The portion below AC could be removed without disturbing the equilibrium of the remainder; hence the weight of the portion BC supports the weight of AB on the inclined plane.

The weights of these portions of chain are proportional to their lengths;

$$\therefore \frac{P}{Q} = \frac{BC}{AB}.$$

He now considers a load Q supported on an inclined plane by a weight P hanging over a pulley at B.

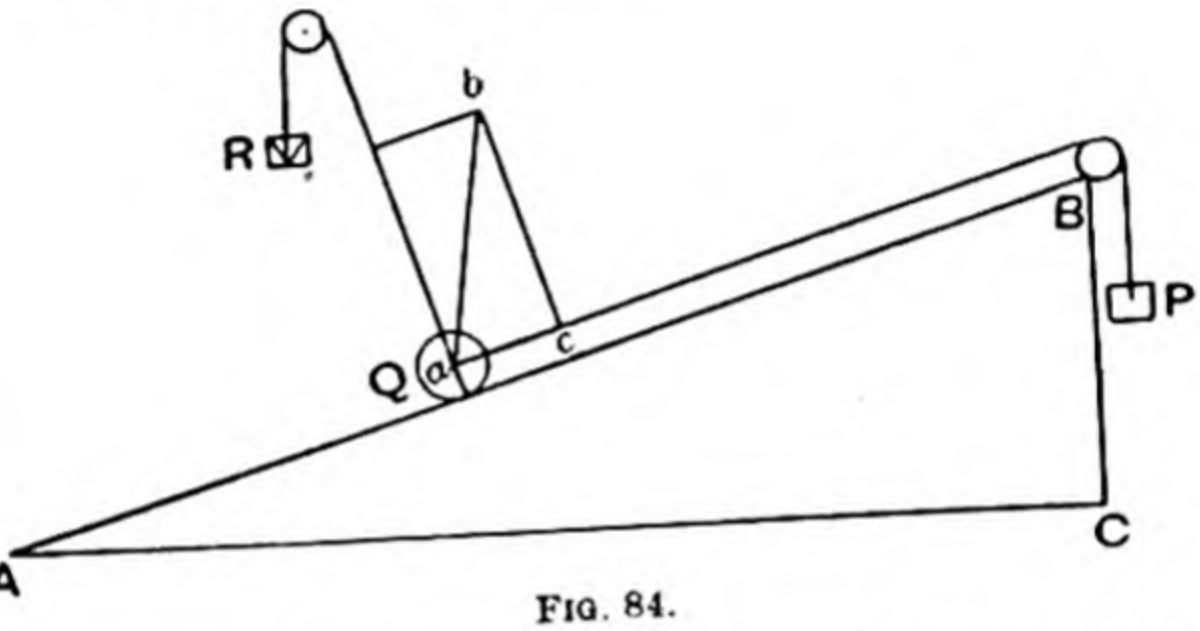



FIG. 84.

He notes that it is not necessary for Q to be on the plane, if the pressure of the plane is represented by a string at right angles to AB, which passes over a pulley and supports a weight R, so that the plane may now be removed.

Draw a vertical line *ab* to represent the load Q on the plane, and draw *bc* perpendicular to *a*B. By his previous result,

$$\frac{P}{Q} = \frac{BC}{AB},$$

and from similar triangles $\frac{BC}{AB} = \frac{ac}{ab}$; $\therefore \frac{P}{Q} = \frac{ac}{ab}$

$\therefore$ *ac* represents P on the same scale as *ab* represents Q.

The proposition was first stated correctly by Newton (1642-1727), who deduced it from dynamical considerations.

Daniel Bernouilli (1700-1782) attempted to furnish a geometrical demonstration by assuming, as an axiom, that the resultant of two equal forces at right angles must bisect the angle between the forces.

**Triangle of Forces. Lami's Theorem.**

If three forces P, Q, R acting at a point are in equilibrium, we have shown that they may be represented in magnitude and direction by the sides of the triangle OAC taken in order.

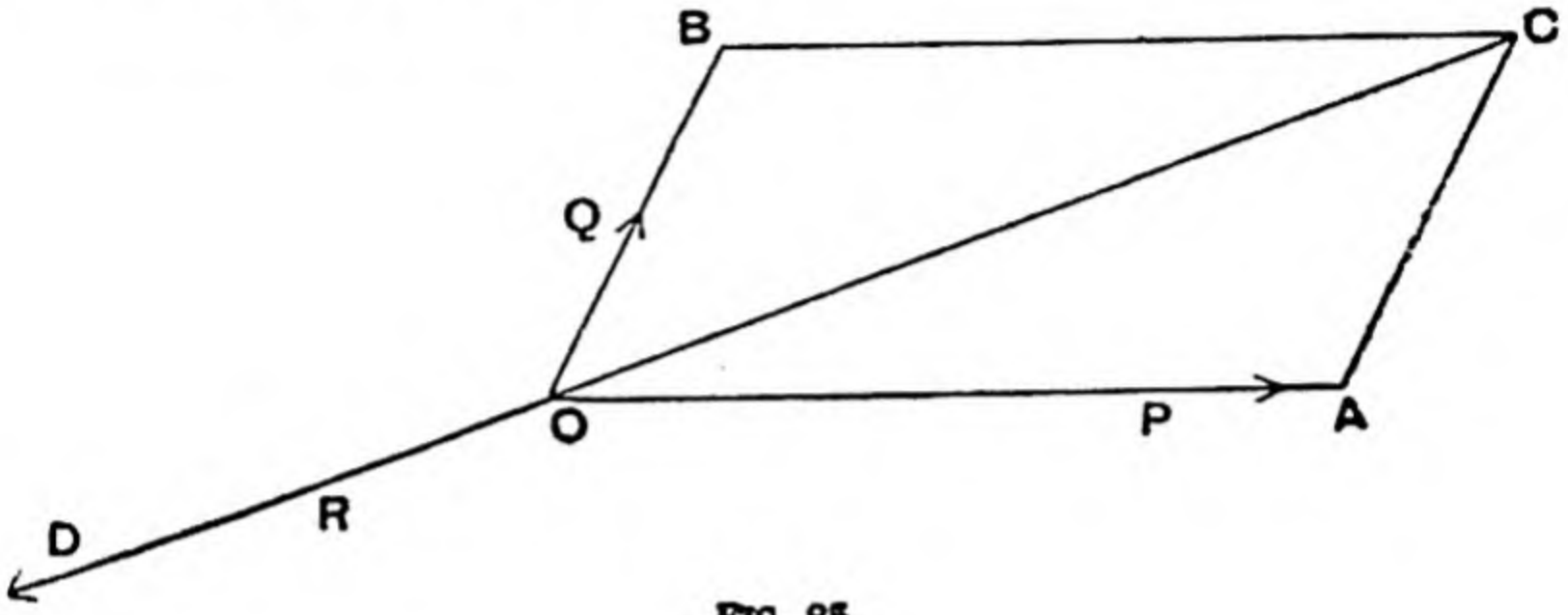



FIG. 85.

Now, by Trigonometry,

$$\frac{OA}{\sin(\hat{OCA})} = \frac{AC}{\sin \hat{COA}} = \frac{CO}{\sin \hat{CAO}}.$$

But $$\sin \hat{OCA} = \sin \hat{BOC} = \sin \hat{BOD} = \sin \hat{QR},$$

since $\hat{BOC}$ and $\hat{BOD}$ are supplementary.

Also $$\sin \hat{COA} = \sin \hat{AOD} = \sin \hat{PR}$$

and $$\sin \hat{CAO} = \sin \hat{BOA} = \sin \hat{PQ}.$$

The sides OA, AC, CO are proportional to the forces P, Q, R;

$$\therefore \frac{P}{\sin \hat{QR}} = \frac{Q}{\sin \hat{PR}} = \frac{R}{\sin \hat{PQ}},$$

**i.e. If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.**

This theorem was first stated by Father Bernard Lami (1687).

EXAMPLE. A weight of 20 lbs. is supported on a smooth plane inclined at $\alpha$ to the horizontal by a force P parallel to the line of greatest slope.

If the plane is smooth, it exerts no force in the direction AB; hence its action is entirely at right angles to its surface.

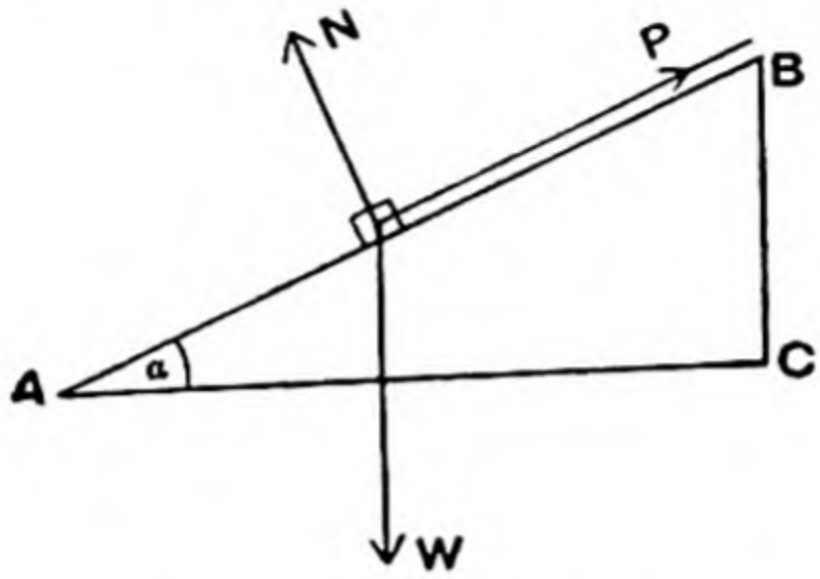



FIG. 86.

Let N be the normal pressure of the plane on the body. The body is in equilibrium under the action of three forces; hence, by Lami's Theorem,

$$\frac{P}{\sin(180-\alpha)} = \frac{W}{\sin 90} = \frac{N}{\sin(90+\alpha)},$$

*i.e.*

$$\frac{P}{\sin\alpha} = \frac{W}{1} = \frac{N}{\cos\alpha};$$

$$\therefore\ P = W\sin\alpha. \quad N = W\cos\alpha.$$ (See p. 60 and p. 87.)

## EXAMPLES XIV.

Solve graphically or by calculation as is most suitable.

1. Find, by Lami's Theorem or otherwise, what horizontal force will support a weight of 50 lbs on a smooth plane inclined at 40° to the horizontal.

2. A weight of 100 lbs is suspended by two ropes inclined to the vertical at angles of 30° and 80° respectively. Find the tension in the ropes.

3. A piece of wire 2 ft. long is stretched between two points in the same horizontal line. When a weight of 100 grams

hangs at its middle point, it sags so that the mid point is 0·4″ below its original position. Find the tension in the wire.

4. Two forces, each equal to 1500 lbs. weight, act at an angle of 160° to one another. Find what force is required to maintain equilibrium.

5. A body weighing 4 lbs. is placed on a smooth plane inclined to the horizontal at 30°. A string is fastened to the body and to a nail above the top of the inclined plane, so that it makes an angle of 20° with the plane. Find the tension in this string.

6. The ends of three rods OA, OB, OC are jointed together at O in one plane.

$$\hat{AOB} = 120°, \quad \hat{BOC} = 100°, \quad \hat{COA} = 140°.$$

If CO is pushed towards O with a force of 10 lbs. wt., find what thrusts AO and BO must exert if equilibrium is not destroyed.

7. The hood of a car is opened by two rods which are jointed together—one rod thrusts on the car, the other on the hood. If the joint is pushed with a force of 10 lbs. wt. when the rods are inclined at 160° to one another, find what thrusts the rods will exert, assuming these forces to be equal to one another.

8. A cord AB, 14 ft. long, is fastened at A and B to two points on the same level 11 ft. apart. A weight of 4 lbs. is fastened to the cord at C, 4 ft from A, and a weight of W lbs. is also fastened at D, 5 ft. from B.

If C is 3 ft. below the line AB, find the value of W.

9. If, in Qu 8, the tension in AC is $T_1$; in CD, $T_2$; in DB, $T_3$; and if these lines make angles $\alpha$, $\beta$, $\gamma$ respectively with the horizontal, prove that

$$T_1 \cos\alpha = T_2 \cos\beta = T_3 \cos\gamma.$$

10. A weight of 20 lbs. is supported in a vertical plane by two strings AB, AC, fastened to two fixed points B and C, so that AB and AC are inclined at 30° and 40° respectively to the vertical.

Resolve the forces acting at A in a direction at right angles to AB, and hence find the tension in AC.

Find the tension in AB by a similar method.

**Three Forces acting on a Rigid Body.**

**If three forces acting in one plane on a rigid body are in equilibrium, their lines of action must meet in a point (or be parallel).**

Suppose a flat board lying on a horizontal table to be acted upon by two forces P and Q at the points A and B.

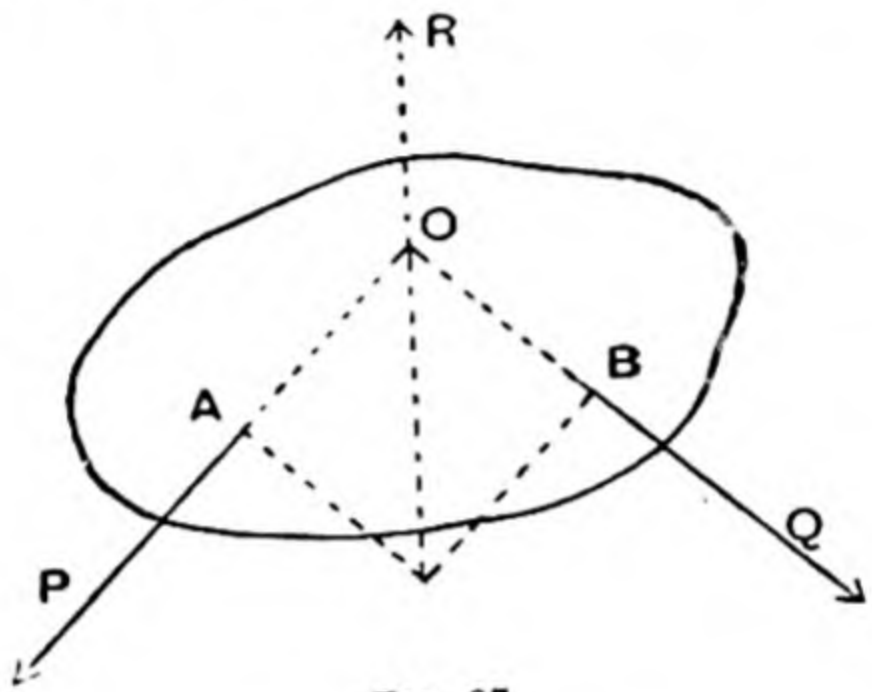



FIG. 87.

Prcduce the lines of action of P and Q until they meet at O. The effect of P and Q will not be altered, if we assume them both to act at O, instead of at A and B.

This is an axiom from experience, and is known as the principle of the Transmissibility of Force.

The resultant of P and Q, acting at O, is a force passing through O, by the parallelogram of forces. Therefore the equilibrant of P and Q is a force passing through O, and it may be supposed to act at any point in the line of the diagonal of the parallelogram whose adjacent sides represent P and Q. If two of the forces are parallel, we have seen in Chapter I. that the equilibrant is a force parallel to the given forces. Therefore the lines of action of the **three forces are either** parallel or meet in a point.

The truth of this theorem is otherwise clear from a consideration of the moments of the forces. If P and Q meet at O, the moment of each of these forces about O is zero, so that if R did not pass through O, its moment about O would not be zero, and the body would not be in equilibrium, under the action of P, Q and R.

**Pressure on the Axle of a Pulley.**

Suppose the cord to be in contact with the surface of the pulley along the arc ACB.

If we neglect the friction at the axle of the pulley, we may assume the tension in the cord at A to be the same as at B.

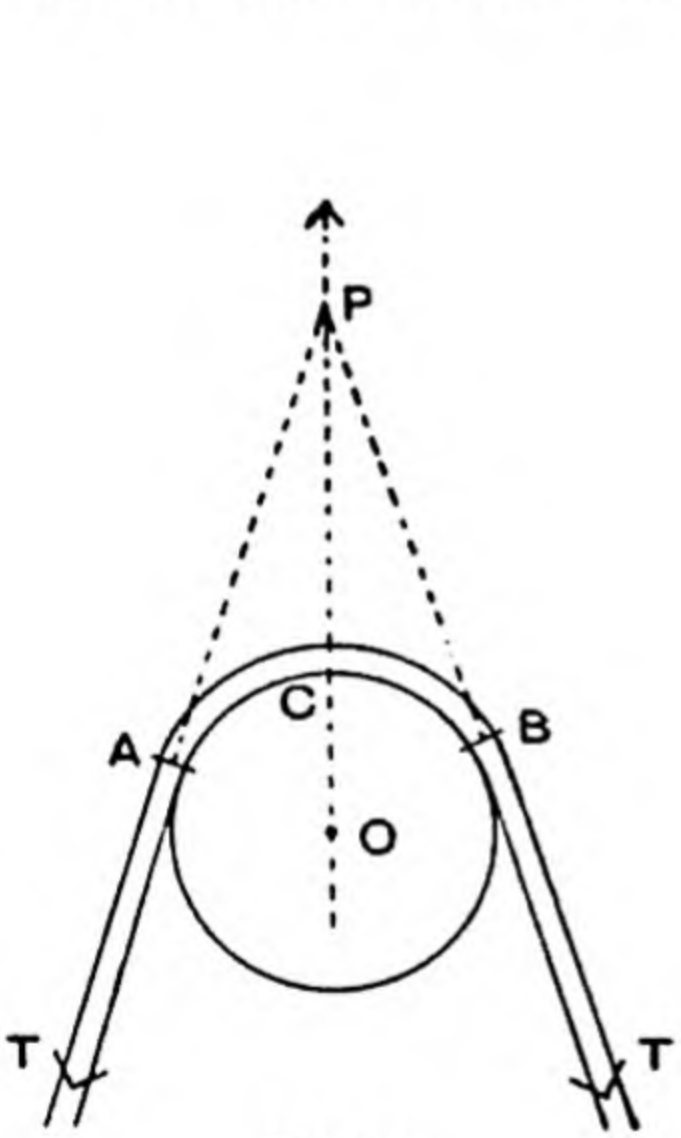



FIG. 88.

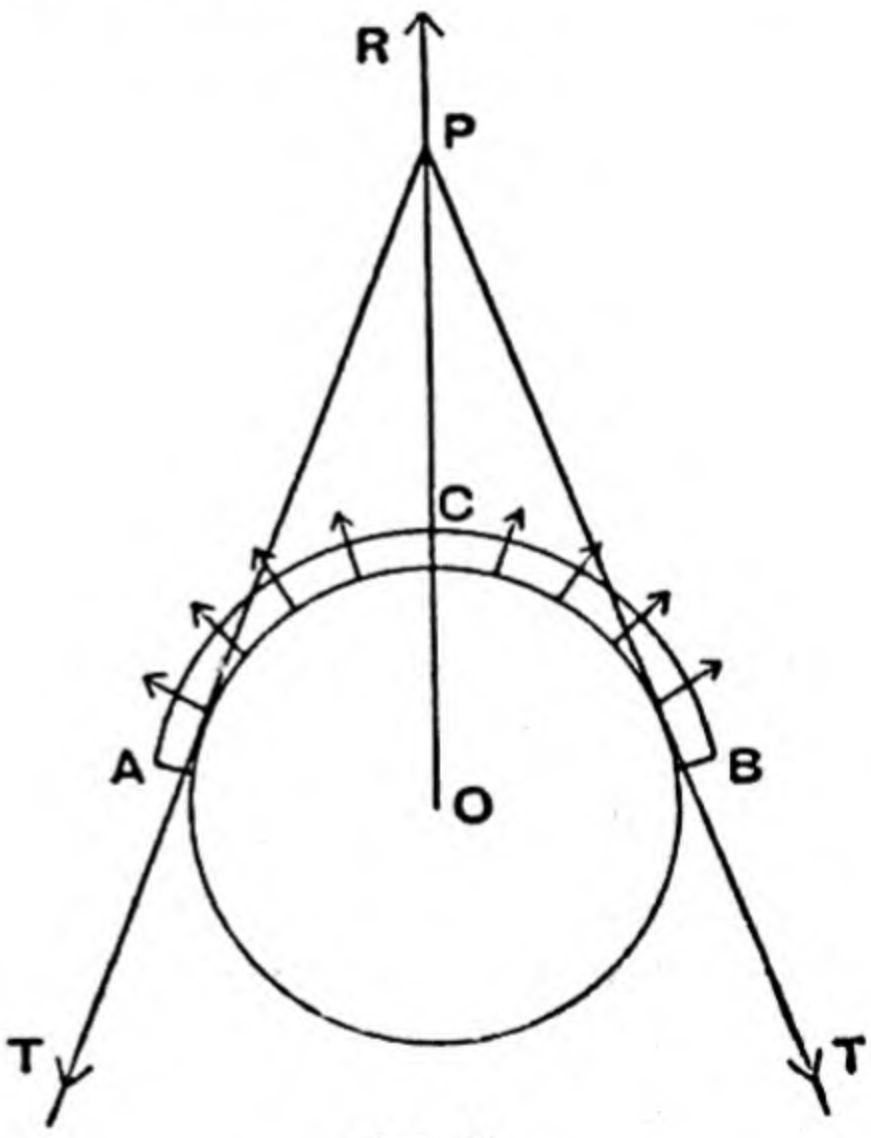



FIG. 89.

The portion ACB of the cord in contact with the pulley behaves as if it were firmly fixed to the pulley, and may for the instant be considered as part of the pulley. This is in equilibrium under the action of T at A, T at B, and the pressure of the axle on the pulley. These forces must meet at a point. Therefore the action of the

axle on the pulley is along OP, and the pressure of the pulley on the axle is therefore along PO. (Fig. 88.)

It is instructive to look upon this in another way.

Consider the equilibrium of the portion ACB of the cord. (Fig. 89.)

This portion, in contact with the pulley, though flexible, does not change its shape, and may be considered as a rigid body.

It is in equilibrium under the action of the pulls at A and B due to the rest of the cord and the pressure of the pulley along the arc ACB. Let the resultant of these various pressures be R. Then the three forces T, T and R must meet at a point P where the tangents intersect. Therefore R acts along OP.

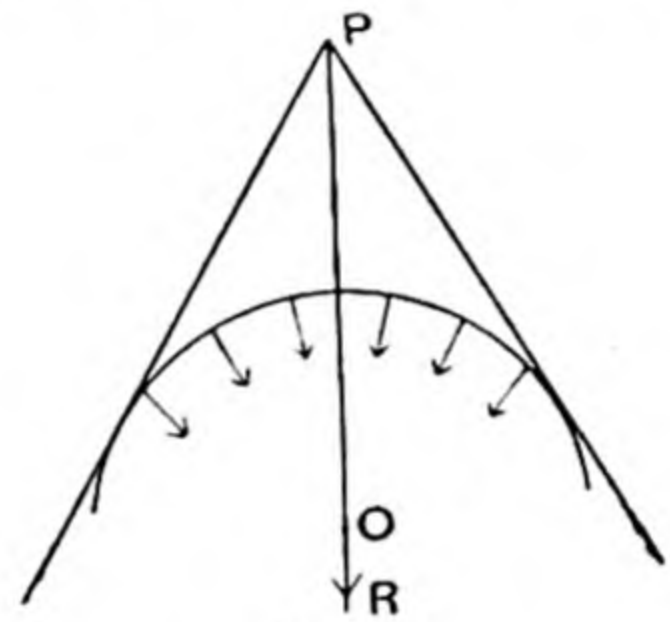



FIG. 90.

Therefore the resultant of the pressures produced by the cord on the surface of the pulley, which is also the force R, acts on the pulley along PO. (Fig. 90.)

EXAMPLE. A uniform board AB of weight W rests with its end A against a smooth vertical wall, and is supported by a string fastened to B, and to a point D in the wall vertically above A. To determine the position of D if AB makes an angle $\alpha$ with the wall.

Since the wall is smooth, the direction of its pressure N on the board is at right angles to the wall; this meets the line of action of W at O.

O is not a point in the board, but the equilibrium of AB would not be affected if we imagine O to be rigidly connected to the board by a light rigid rod.

The line of action of the tension must, therefore, pass through O, and will meet the wall at D, which is thus determined.

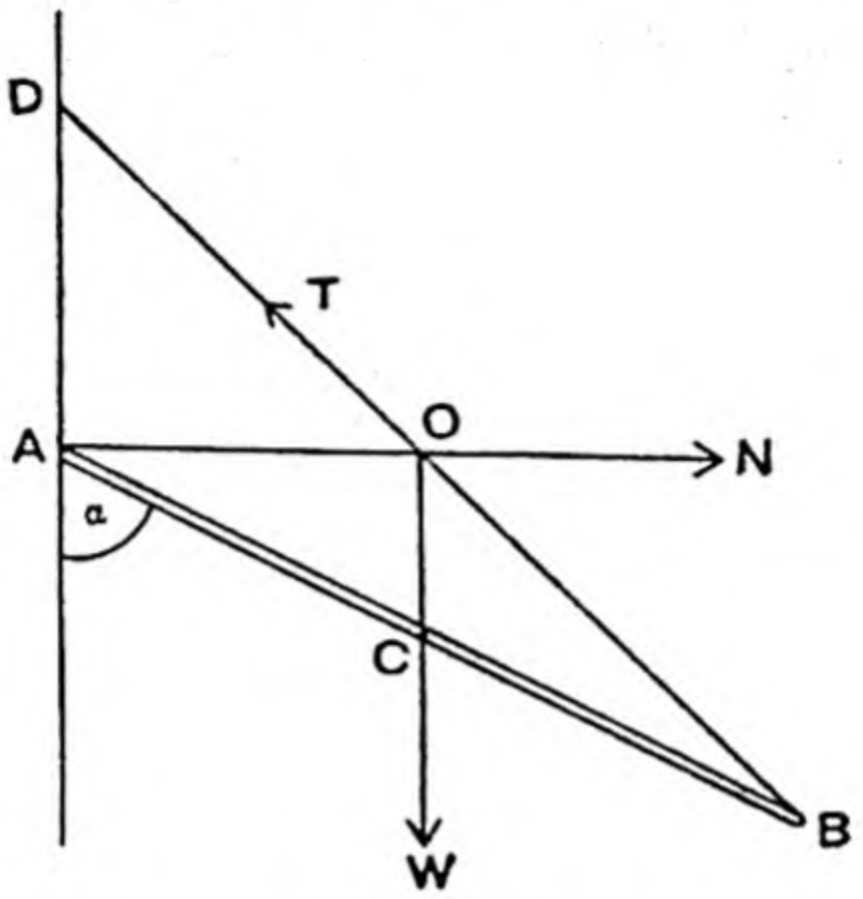



FIG. 91.

A triangle of forces may now be drawn for the three forces acting at O.

## EXAMPLES XV.

**1.** A light rod BC, whose weight may be neglected, is supported by two unequal strings fastened to its ends B and C, and to a point A. A weight is attached to the rod at P. Draw figures to show the position of P (1) when BC is horizontal; (2) when the tensions of the strings are equal.

**2.** A uniform rod AB, 4 ft. long, is hinged to a vertical wall at A, and is supported in a horizontal position by a string CD fastened to the rod at C, 3 ft. from A, and to the wall at D, a point 4 ft. above A. If the weight of the rod is 10 lbs., find graphically the tension of the string and the magnitude and direction of the reaction of the hinge.

**3.** A rod AB, 5 ft. long, weighing 5 lbs., is hinged to a wall at A, and is pulled by a horizontal force P at B, so that the rod makes 40° with the vertical. Find, graphically, the direction and magnitude of the pull on the hinge.

**4.** A uniform rod AB rests with the end A against a smooth vertical wall at A, and passes over a smooth horizon al rail which is parallel to the wall and 1 ft. from it. If the rod is in a plane perpendicular to the wall and makes an angle of 30° with the horizontal, find graphically the length of the rod.

**5.** A sphere is hung from a point in a vertical wall by a string equal in length to the radius of the sphere. If the sphere weighs 5 lbs., find the pressure against the wall and the direction and tension of the string.

**6.** One end of a uniform ladder, weighing 80 lbs., rests against a smooth vertical wall 12 ft. from the ground; the other end is on the ground, 8 ft. from the wall. Find the magnitude and direction of the pressure on the ground.

**7.** A rectangular picture, weighing 4 lbs., is hung from a smooth peg by a cord which is 3 ft. long, and has its ends fastened to the top horizontal edge of the picture 20 in. apart. Find the tension in the string and the pressure it produces on the peg.

**8.** A rod AB, 3 ft. long, weighs 2 lbs. Its end A rests against a smooth wall, and the rod is supported in a vertical plane, so that it makes 30° with the horizontal, by a string fastened to B and to a point C in the wall vertically above A. Find, by drawing and by calculation, the length of BC, and determine the tension of the string.

**9.** A rod AB, 80 cms. long, is hinged at A, and is kept at 70° from the vertical by a force at B in the same vertical plane as the rod, which makes an angle of 50° with the vertical. Find the direction and magnitude of the action at the hinge if the rod weighs 5 lbs.

**10.** A uniform seat, 4 ft. long, is hinged at one end A to a beam, and is supported in a horizontal position by a light rod fastened to a point 6 in. from the other end of the seat and to a point 4 ft. vertically below A. If a boy, weighing 10 stone, sits at the middle of the seat, find the additional thrust he causes at the hinge and in the rod.

**11.** A rod, $2\frac{1}{2}$ ft. long, lies inside a fixed smooth hemispherical bowl with its lower end 6 in. above the level of the lowest point of the bowl. If the radius of the bowl is $1\frac{1}{2}$ ft., and the rim horizontal, find the position of the c. of g. of the rod.

**12.** ABCD, a rectangular lamina of weight W, can turn in a vertical plane round A, its highest point. The mid point of the lower edge CD rests on a smooth peg at E, so that DC makes an angle $\alpha$ with the horizontal. If AB = 8 in. and BC = 6 in., prove that the reaction at A is along the diagonal and that its magnitude is $\frac{5}{4}$ W sin $\alpha$.

**13.** A rod AB, whose weight is 54 lbs.. is supported by ropes AC and BD, 2·2 and 1·7 ft. long, fastened to two points C and D in the same horizontal line, 4·8 ft. apart, making angles of 55° and 40° respectively with the vertical. Determine the position of the C. of G. of the rod and the tensions of the ropes. (Army.)

**14.** A uniform rod AB, weighing 2 lbs., is 8 ft. long and can turn freely about A. It is held in a horizontal position by a string in a vertical plane fastened to B and inclined at 45° to the rod. If a body, weighing 2 lbs., is tied to the rod 6 ft. from A, find the tension in the string and the reaction at A.

**15.** A uniform plank, 10 ft. long, weighing 40 lbs., has one end on the ground and the other resting on a smooth plane inclined at 40° to the horizontal, the plank being in the same plane as a line of greatest slope of the plane. If the plank is inclined at 20° to the horizontal, find the total reaction of the ground on the lower end, and determine how much of this is due to friction.

**16.** A smooth, thin rod, 14 in. long, weighing 4 oz., hinged to a wall at A, can move in a vertical plane perpendicular to the wall. A piece of string, 1 ft. long, is tied to a point on the wall, 15 in. vertically above A, and a small smooth ring at the other end of the string is slipped over the rod. Find the position of equilibrium and the magnitude and direction of the reaction at the hinge.

**17.** AB is a rod, 4 ft. long, which can turn in a vertical plane round an axis at its C. of G., 1 ft. from B. Two strings OA, OB, each 4 ft. long, are knotted at O, and a weight of 5 lbs is suspended from O. Find graphically the angle AB will make with the horizontal, and find also the tension in the strings.

**18.** Three rods, AB = 3 ft., BC = 5 ft., CA = 4 ft., form a triangle, loosely jointed at the corners, which lies on a hori-

zontal plane. An elastic string is fastened to D in AB, 1 ft. from A, and to E in AC, 3 ft. from A. If the tension in this string is lbs. wt., find the magnitude and direction of the reactions at A, B and C.

**19.** A rigid body is acted upon by three forces which act along the sides of a triangle ABC in the directions AB, BC, CA. If the forces are proportional respectively to the sides of the triangle along which they act, prove that they are equivalent to a couple and that the body will therefore rotate.

**20.** A very small ring is free to slide on a smooth circular wire of radius $a$, fixed in a vertical plane. It is attached by a string of length $l$ (less than $\sqrt{2} \cdot a$) to a point on the wire at the same height as the centre. If W be the weight of the ring, show that the tension of the string is $\dfrac{(2a^2 - l^2)W}{a\sqrt{4a^2 - l^2}}$.

**21.** AB and AC are two light rods, each 4 ft. long, both of which can turn in a vertical plane round A. AC makes 45° with AB, which is horizontal (C above AB). A light string joins D, the mid-point of AC, to E in AB, 3 ft. from A. If 5 lbs. is hung at B, find the least force at C which will keep the rods in the given position, and find the tension in ED.

**22.** A uniform heavy beam AB, 8 ft. long, weighing 50 lbs., is supported by a vertical rope at B, and by another rope, 2 ft. long, fastened to A and to a point C on the same level as B and 9 ft. from it. Find what horizontal force at B is required to keep the beam in the given position.

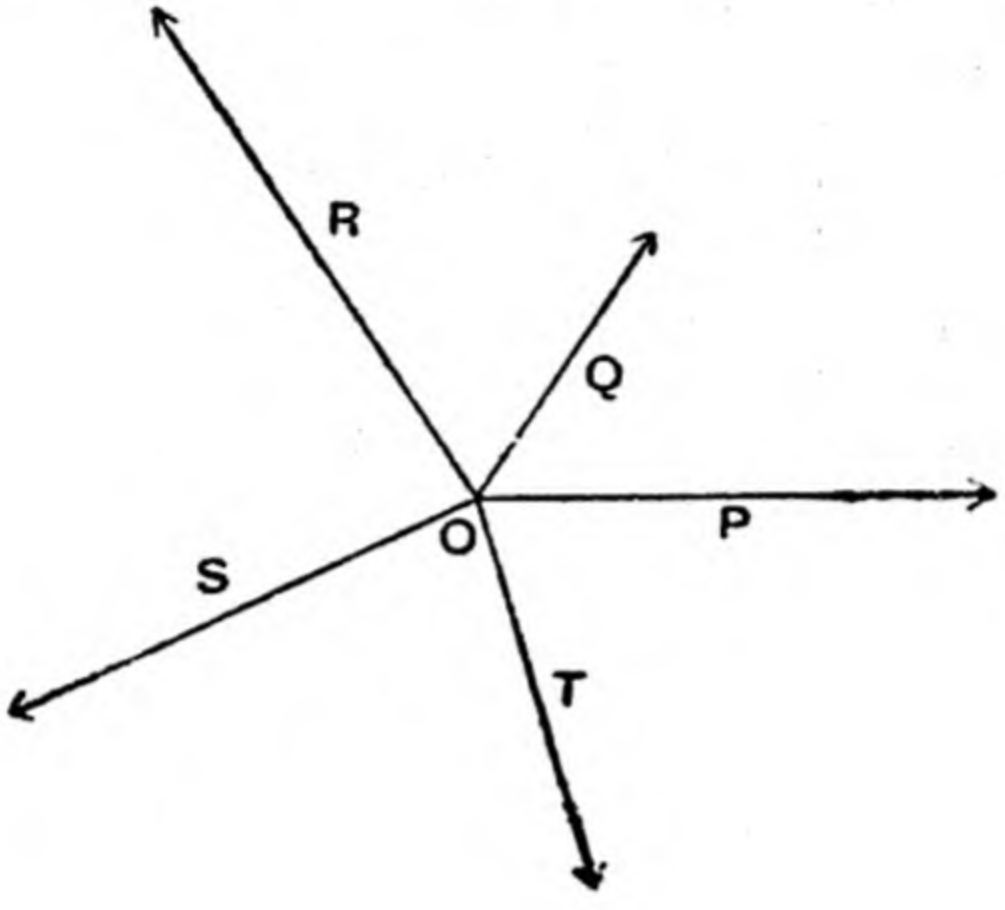
*N.B.*—Most of the questions in Examples III., p. 30, may be solved by the Triangle of Forces.

**Polygon of Forces.**

**If any number of forces acting at a point can be represented in magnitude and direction (but not in position) by the sides of a polygon taken in order, then the forces are in equilibrium.**

The forces P, Q, R, etc., acting at O, are represented in magnitude and direction by the sides of the polygon ABC...E taken in order, to prove that the forces are in equilibrium.

By the triangle of forces, since AB represents P and BC represents Q, their equilibrant is represented by CA. Therefore AC represents their resultant.



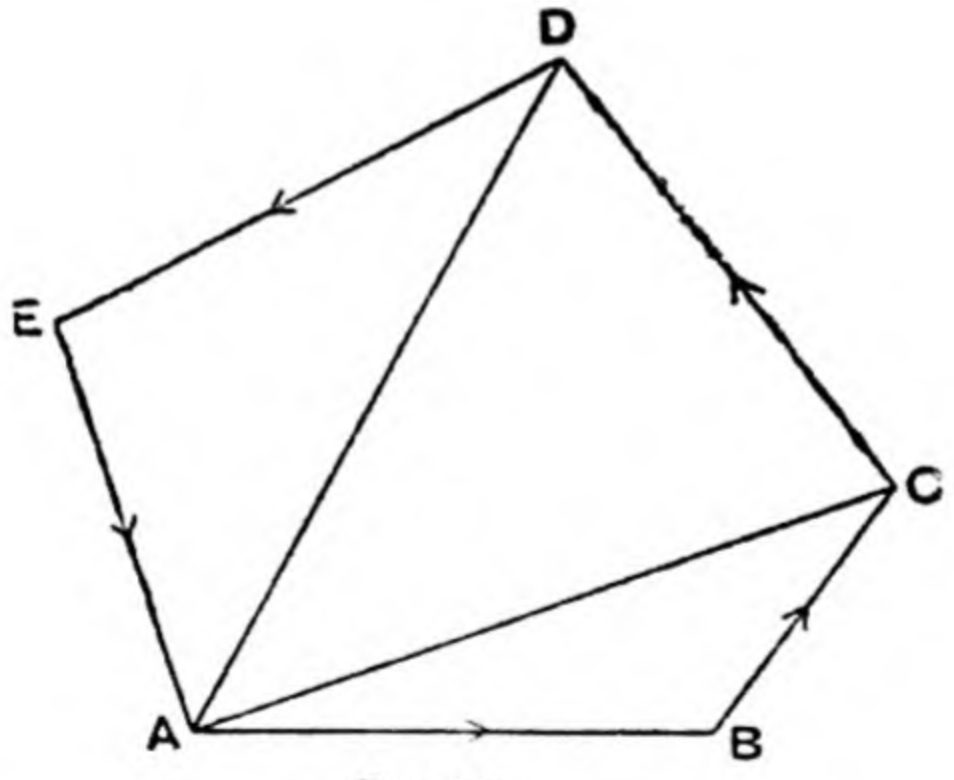




FIG. 92.

The resultant of P, Q and R, acting at a point, is the same as the resultant of the forces represented by AC and CD, and is therefore AD.

Finally, the resultant of the forces of P, Q, R, S, T, acting at a point, is represented by the resultant of forces represented by AD, DE, EA, but these are in equilibrium by the triangle of forces.

When we say that the sides of the polygon must be "taken in order," we mean that we must travel continuously round the boundary of the figure, but there is no necessity for the forces to be considered in the order P, Q, R, etc.

For instance, the polygon ABCDE (Fig. 93) represents the forces P, Q, R, etc., in the order P, R, S, Q, T.

Draw a figure to show that if any polygon were drawn with sides parallel to the sides of ABCDE, these sides would not necessarily represent the forces P, Q, R, etc.

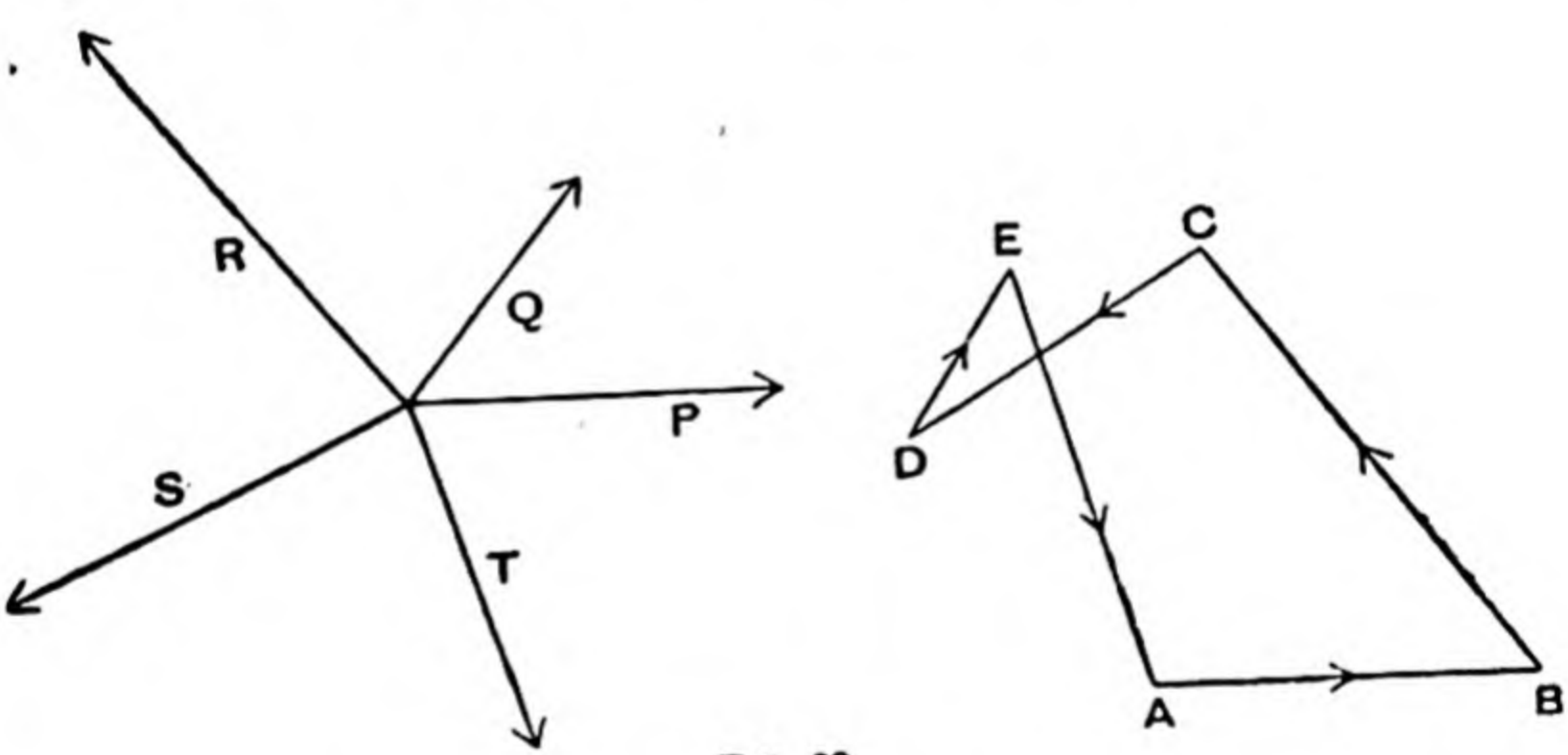



FIG. 93.

The polygon of forces may be usefully employed to find graphically the *resultant* of a number of forces acting at a point. It is clear that if we were given the forces P, Q, R, S, acting at O, their resultant would be represented in magnitude and direction by the side AE of the polygon.

**Results by Calculation.**

The resultant of a number of forces acting at a point may be calculated by resolving all the forces in two directions at right angles to one another.

To find the resultant of the forces acting as represented in the figure.

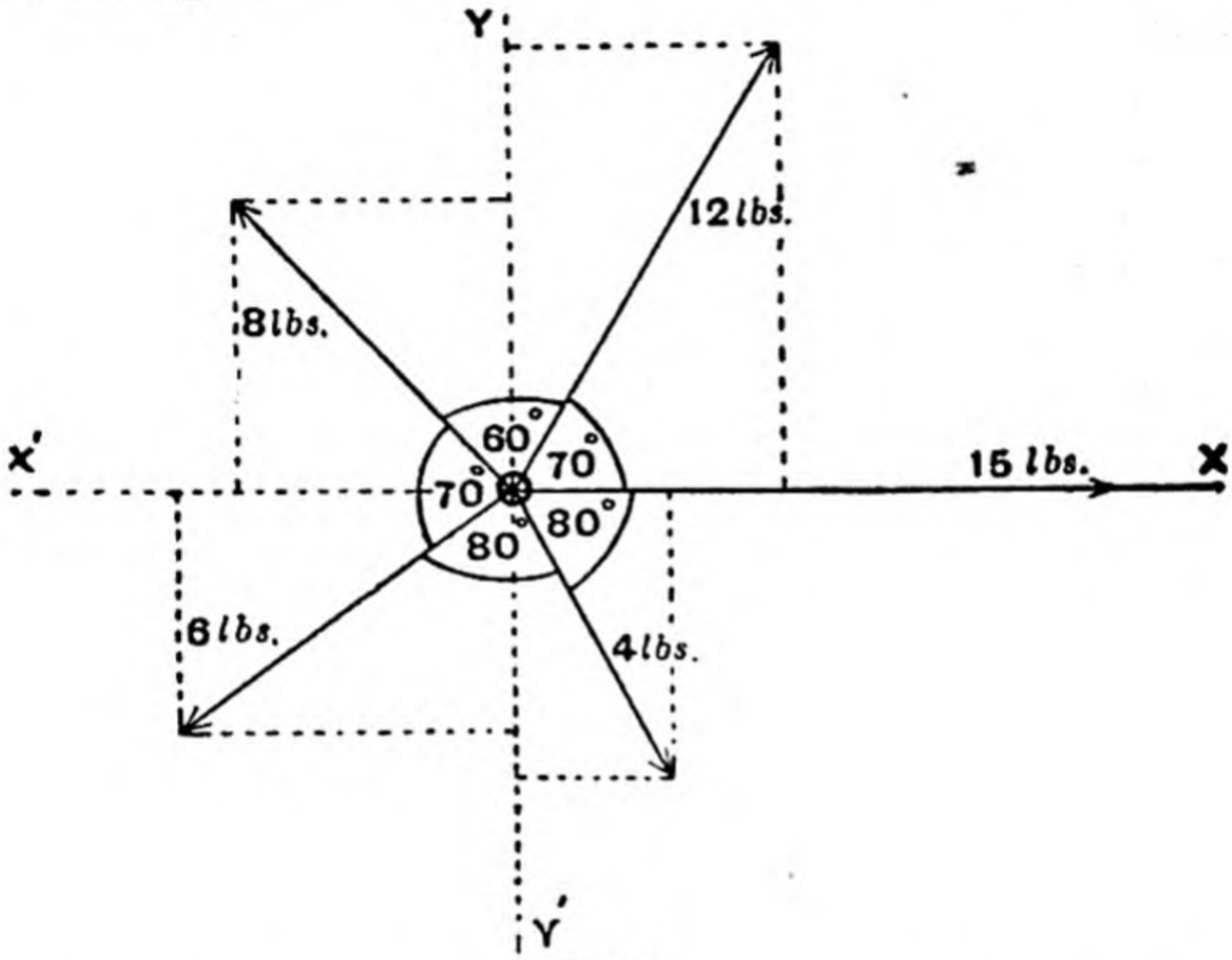



FIG. 94.

Select any suitable direction OX (generally the direction of one of the forces, say the line of action of the 15 lbs. wt.) and another direction OY through O at right angles to OX.

Resolve all the forces in these two directions.

The system is equivalent to :

$$15 + 12\cos 70 + 4\cos 80 = 19{\cdot}8 \text{ along OX},$$
$$8\sin 40 + 6\cos 20 = 10{\cdot}8 \quad ,, \quad \text{OX}',$$
$$12\sin 70 + 8\cos 40 = 17{\cdot}3 \quad ,, \quad \text{OY},$$
$$6\sin 20 + 4\sin 80 = 5{\cdot}98 \quad ,, \quad \text{OY}',$$

that is 9·0 along OX and 11·4 along OY.

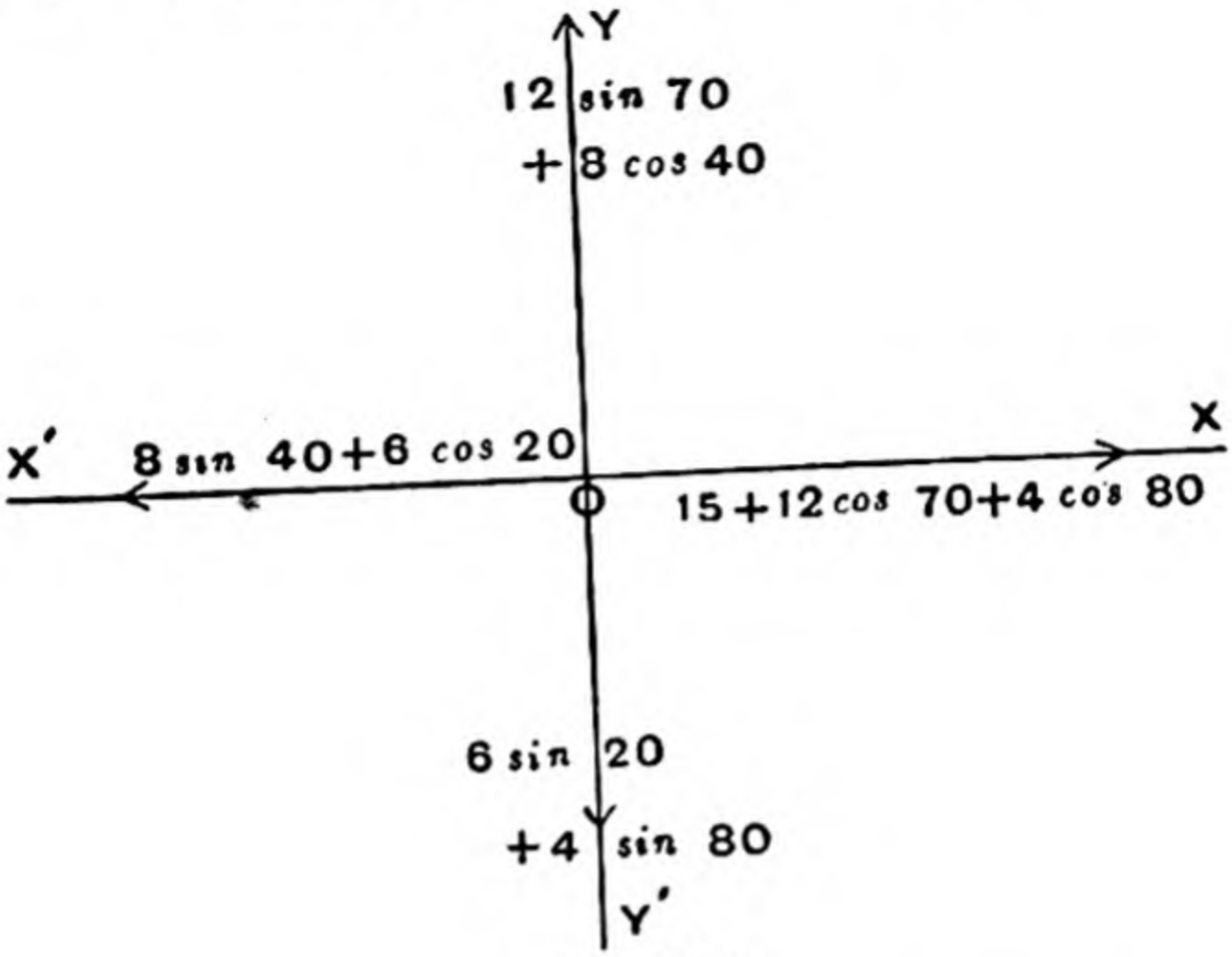



FIG. 94(a).

If R is the resultant of these two forces, we have

$$R^2 = (9{\cdot}0)^2 + (11{\cdot}4)^2 = 211\ ;$$

$$\therefore\ R = 14{\cdot}5 \text{ lbs. wt. and } \tan\theta = \frac{11{\cdot}4}{9{\cdot}0}\ ;$$

$$\therefore\ \theta = 51^\circ\,48'.$$

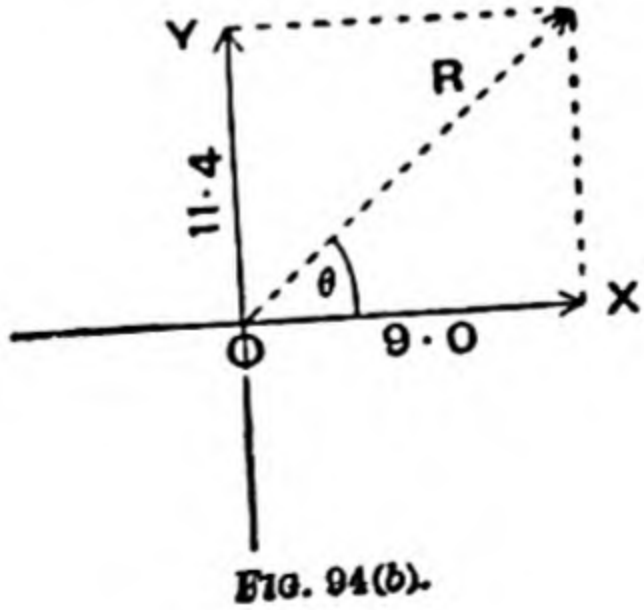



FIG. 94(b).

## EXAMPLES XVI.

**1.** Find, by calculation and by drawing, the resultant of four forces, each equal to 10 lbs. wt., acting at a point O, and making angles of 50° with each other.

**2.** Three rods, OA, OB, OC, are jointed at O. OA is horizontal, OA makes 50° with OB, and OC makes 60° with OB on the other side. If there is a thrust of 20 lbs. wt. along AO, and a tension of 15 lbs. in OB and in OC, find the direction and magnitude of the resultant thrust on O.

**3.** Two cords, CA 2 ft. and CB 3 ft., are fastened to fixed points A and B, 4 ft. apart, in a horizontal line. A weight of 200 grams hangs from C. If a horizontal force P of 80 grams wt. is now applied to C in the direction A to B, find graphically the tensions in CA and CB, and find also what is the greatest value of P consistent with the cords remaining taut.

**4.** Draw on squared paper a rectangle 10 units by 8. Take the longer edge as the X axis, and mark the point O, (3, 2). Join this point to the four angular points. Suppose forces proportional to these lines to act at O towards the corners, and find the magnitude and direction of their equilibrant.

**5.** Four forces acting in a plane at a point O are represented in magnitude and direction by the lines OA, OB, OC and OD.

$$OA = 2'', \ OB = 3{\cdot}5'', \ OC = 1{\cdot}2'', \ OD = 4''.$$

$$\hat{AOB} = 15°, \quad \hat{AOC} = 90°, \quad \hat{AOD} = 221°.$$

An inch represents 10 lbs. wt. Draw a diagram to scale, and from it find the magnitude of the resultant OF and the angle AOF. (Army.)

**6.** Find graphically, on squared paper, the resultant of forces of 20 lbs. wt. E., 23 lbs. wt. S.W. and 17 lbs. wt. 30° S. of E., all acting at the same point. What forces N.E. and S. will be equivalent to this resultant?

**7.** Three strings, OA, OB, OC, knotted together at O, support a weight of 10 lbs. hanging from O. OA is inclined to the vertical through O at 30° on the left side, OB and OC on the right of this vertical make 50° and 35° respectively with it. OC passes over a smooth pulley and supports a weight of 2 lbs. Find the tensions in OA and OB.

## Laboratory Experiments.

**1.** Experiment with a crane as given in the text (p. 72).

**2.** Clamp a pulley to an edge of the given board, and use the board as an inclined plane, setting it at about 30° to the horizontal.

Pass a cord over the pulley at the top of the board. Fasten one end to a loaded trolley placed on the inclined plane, and attach a suitable weight at the other end (or pull with a dynamometer), so as to cause the trolley to travel slowly up the plane when given a slight starting impulse. Note the pull required. Find also the pull in the cord when the trolley just runs down.

Assume that the average of these pulls is that under which the trolley would remain at rest if it were frictionless, and from this result and the measured inclination of the board (measure the tangent of the angle), construct a force diagram and find the weight of the loaded trolley.

Find the resultant thrust of the cord on the pulley.

**3.** Measure the length of the given spring (1) when not loaded; (2) when loaded with 150 grams.

Assume that the pull in the spring is proportional to its extension beyond its natural length. Suspend the rod CD from A and B by two vertical threads AC and BD, so that CD is horizontal. Tie a thread to each end of the spring, fasten it to C, and pass the other end through the hook B. Fasten it at E so that the rod rests horizontally while AC and BD are inclined to the vertical, both being taut.

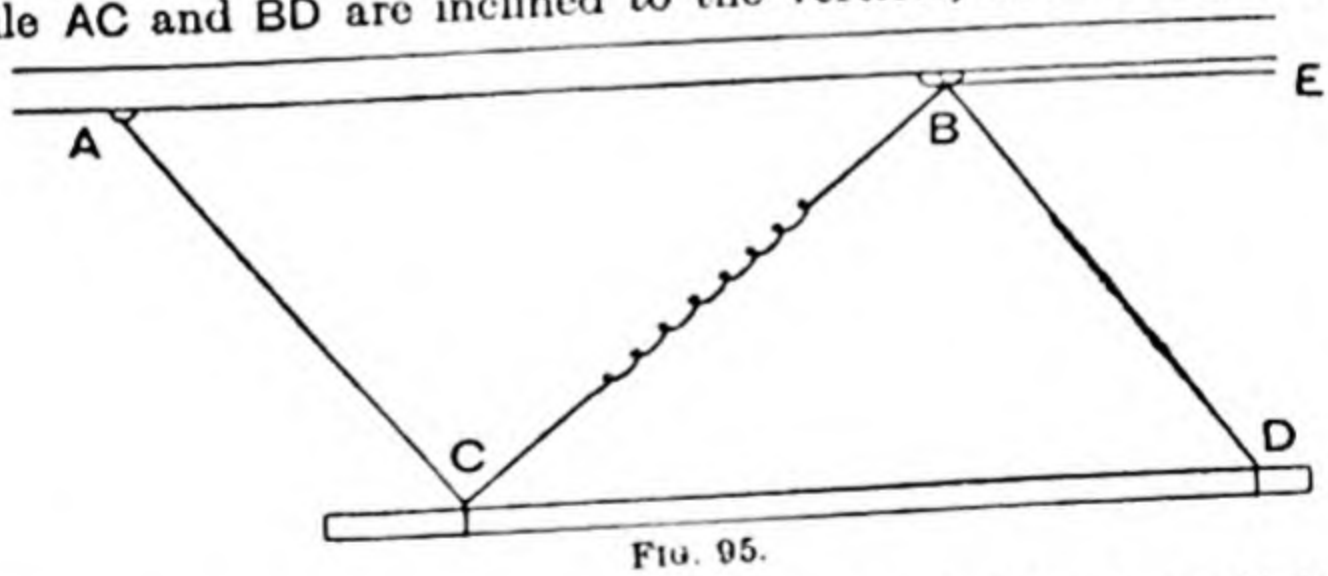



FIG. 95.

Measure the length of the spring, and calculate the pull in CB.

Measure the sides of the triangle ABC. Enumerate the forces on the rod. Note that two of them are parallel, and that one force parallel to CA might be substituted for these two. Draw ABC to scale, and find the weight of the rod. (Army.)

## MISCELLANEOUS EXAMPLES.

### CHAPTERS I.–III.

### A.

**1.** A string ABC hangs vertically from A and has a weight of 7 oz. fastened at B and 4 oz. at C. Find the tension in the parts AB and BC.

**2.** A uniform rod weighing 10 lbs. passes through two fixed rings and is horizontal. The rings are 3 ft. apart, and the lengths of the parts of the rod which project beyond them are 2 ft. and 11 ft. Find the pressure on each ring.

**3.** The arms AB, BC of a light, bent lever, which can turn round B in a vertical plane, are at right angles to one another, and AB = 2BC. A weight of 18 lbs. is suspended from C and 12 lbs. from A. Find the angle AB makes with the horizontal in the position of equilibrium.

**4.** An iron wire, $\frac{1}{8}$ in. in diameter and 12 ft. long, is securely attached to two fixed points on the same level $10\frac{1}{2}$ ft. apart. What weight suspended at the mid point will break it, if it will stand a strain of 75,000 lbs. to the square inch?

**5.** A man weighing 12 stone, standing on the ground, supports a weight of 5 cwt. by means of a block and tackle which has three sheaves in each block. If the lower block weighs 10 lbs., find the pressure the man exerts on the ground (neglect friction).

### B.

**1.** The arms of a false balance are in the ratio 7 : 8, and the weight is put into one scale as often as into the other. Find how many lbs. are gained or lost in each cwt. that is sold in separate lbs.

**2.** A board ABCD of uniform thickness is free to move on a smooth horizontal table about the vertex D, which is attached to the table by a smooth vertical pin. AB = 14 in., AC = 21 in. and AD is parallel to BC. Show that if forces equal to weights of 1 lb. and $1\frac{1}{2}$ lb. act respectively along AB and CA, the board will not move.

**3.** A beam, 15 ft. long, balances at a point one-third of its

length from the thicker end, but when a weight of 15 lbs. is suspended from the other end, the prop must be moved 3 ft. towards it, to maintain equilibrium. Find the weight of the beam.

**4.** A light thread passes over two pulleys A and B, 1 ft. apart, at the same level, and carries a weight of 3 lbs. at each end, and a weight of 2 lbs. fastened to its mid point. Find the depth below the level of AB at which its mid point will rest.

**5.** With a certain machine a load of 1 ton can be raised by an effort of 300 lbs., and the efficiency is 75 %. If with the same machine an effort of 110 lbs. lifts a load of 500 lbs., find the efficiency.

## C.

**1.** An equilateral triangular lamina is suspended by threads fastened to two of its angular points. The direction of the threads produced passes through the C. of G. of the lamina, and one of them is horizontal and has a tension of 1 lb. Find the tension of the other thread and the weight of the lamina.

**2.** A safety valve consists of a circular hole, $\frac{1}{4}$ in. diameter, closed by a plunger attached to a light horizontal hinged bar, 1 in. from the hinge. A weight of 1 lb. slides on the bar. How far from the hinge must it be set if steam is to blow off at a pressure of 160 lbs. to the square inch?

**3.** Draw two axes OX, OY at right angles to one another on squared paper. Join O to the three points A(4, 2), B(1, 5), C(−3, −3). If OA, OB, OC represent the magnitude and direction of three forces acting at O, find their resultant.

**4.** In a Weston pulley block the radii of the two sheaves are $5\frac{1}{2}$ in. and 6 in. respectively. Find the velocity ratio and the efficiency if a pull of 74 lbs. will raise a load of 800 lbs.

**5.** A uniform rectangular column of stone is 8 ft. high, and stands on a base 2 ft. square. If it weighs 3 tons, find what horizontal force applied at the top will just tilt it.

## D.

**1.** A uniform beam, 8 ft. long, weighing 12 lbs., is supported horizontally by two vertical strings 1 ft. and 2 ft. respectively from the ends. Find the tension in the strings, and find what

weight must be placed at one end of the beam to make the tensions equal.

2. AB is a light rod turning about its mid point C. AD is a heavy uniform rod jointed to AB at A, while D is fastened to B by a string BD, so that ADB is a right angle. Draw a figure showing the position of equilibrium, and find the direction in which the rod AD acts on AB at A.

3. ABC is a uniform heavy lever, of which B is the fulcrum, the arms AB, BC being inclined at 120° to one another. If, when the lever is left to itself in a vertical plane, the arm AB is horizontal, find the ratio BA : BC.

4. Find the velocity ratio of a screw-jack in which the pitch of the screw is $\frac{5}{8}$ in. and the length of the lever to which the effort is applied is 2 ft. If the efficiency is 30 %, find what force will overcome a resistance of 1500 lbs.

5. Two smooth inclined planes of the same height are placed back to back. A weight of 10 lbs. placed on one whose slope is 40° is supported by a string over a pulley at the top fastened to a weight W on the other plane. If the slope of this plane is 30°, find W.

## E.

1. A uniform rod AB, 4 ft. long, weighing 1 lb., is placed on a table with the end A projecting over the edge. If 4 ozs. is hung from A, find how far the rod may project without turning over.

2. AB, CD are two equal light rods which can turn round A and C respectively, and are connected by a vertical string PQ. A weight of 5 lbs. is placed at D, and the rods are supported

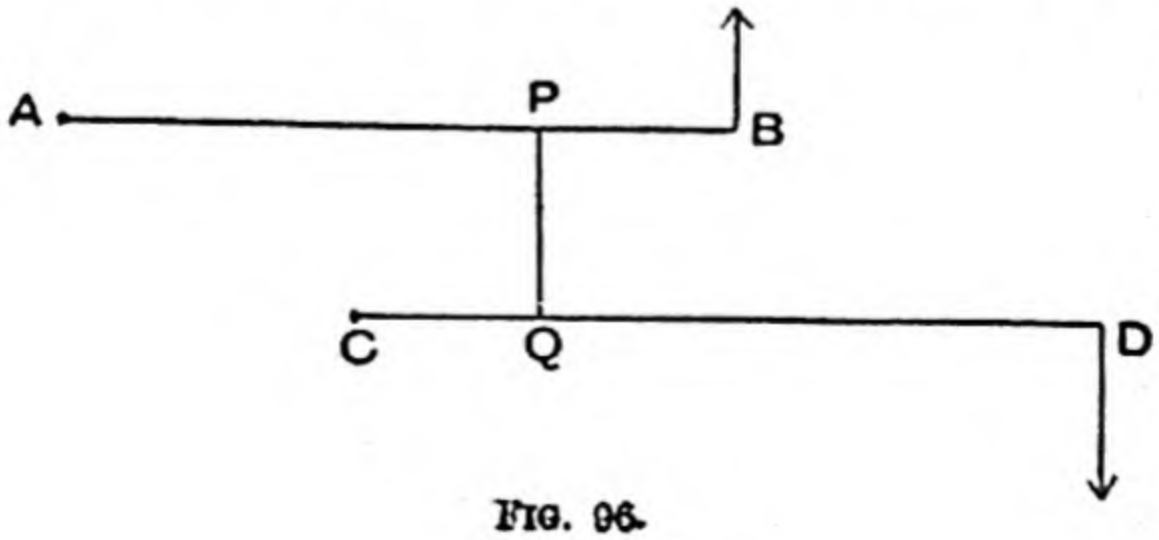



FIG. 96.

in a horizontal position by a vertical force at B. If CQ = 5 in., AP = 12 in., AB = CD = 18 in., find the reactions at A and C, and the force at B.

**3.** A uniform rod AB, 4 ft. long, rests in a plane at right angles to a smooth wall, with one end resting against the wall, and it is supported by passing over a smooth rail parallel to the wall. If the rod is inclined at 40° to the vertical, find how far the rail is from the wall.

**4.** A rectangular board ABCD is fixed in a vertical plane with its top edge AB horizontal, and four pulleys are fastened at the corners. AB = 4 ft., AD = 2 ft. O is a point on the board 1 ft. from DC and 1 ft. from AD. Four strings passing over the pulleys are knotted together at O, and carry weights at their other ends. If the string OA carries 5 lbs., and the string OB 3 lbs., find the weights carried by OC and OD, when O is in equilibrium.

**5.** A uniform rod AB, of weight W, hanging by a string attached to it at A, is drawn from the vertical by a force acting horizontally on B, and equal to one-third the weight of the rod. Find the tension of the string and the angle which it makes with the vertical, when the rod is in the position of equilibrium.

## F.

**1.** Forces of 3, 4, 6 lbs. wt. act at a point parallel to the sides of an equilateral triangle taken in order. Find the magnitude of the resultant and the angle it makes with the force of 3 lbs. wt.

**2.** A uniform heavy rod AB passes under a peg C and over

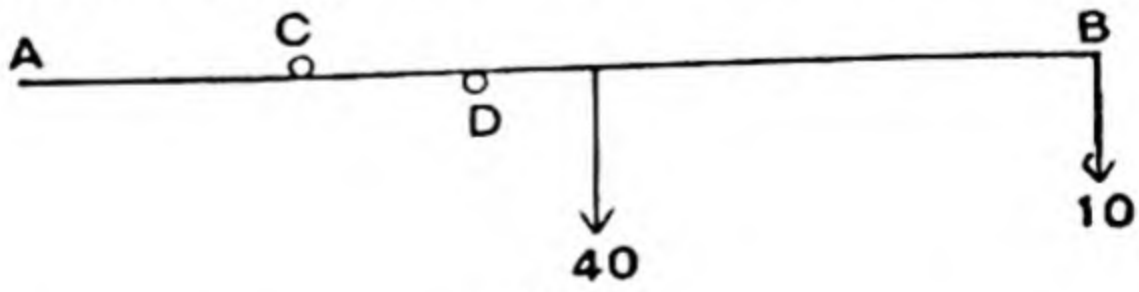



FIG. 97.

a peg D, 2 ft. apart. The rod is 15 ft. long, and the end B projects 9 ft. beyond D. If the rod weighs 40 lbs. and a weight

of 10 lbs. is suspended from B, find the pressures exerted by the pegs.

**3.** If a weight W is suspended from A in Qu. 2, find the greatest value of W if equilibrium is not to be destroyed.

**4.** The diagram represents a stand, of which the base P weighs 25 lbs. and the T piece is made of metal rod which weighs 1·7 lbs. to the foot length. The scale of the diagram is

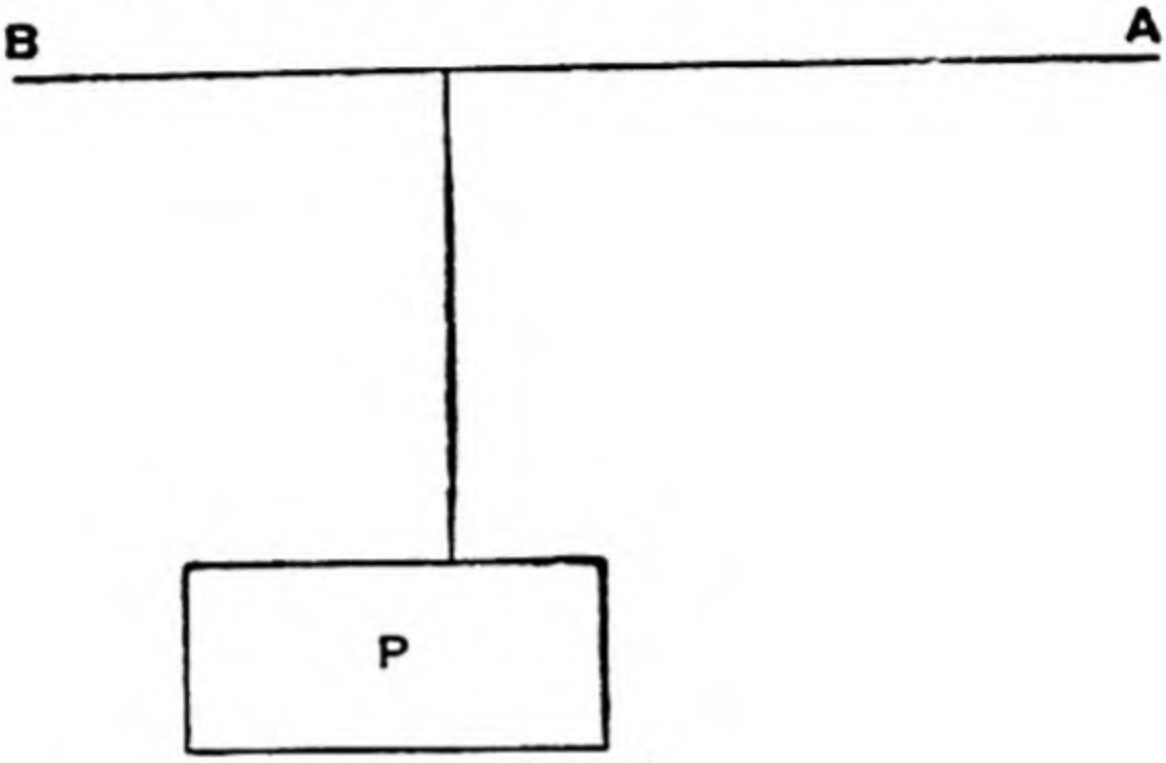



FIG. 98.

1 cm. to 1 ft. Find the greatest weight which can be hung from A without overturning the stand; and if 25 lbs. more than this is hung there, find the least and the greatest weights which can be hung from B. (Army.)

**5.** Three rods, each 3 ft. long, are freely jointed together at their extremities so as to form an equilateral triangle ABC, which is placed on a table. An elastic cord DE is fastened to D in AB, 1 ft. from A, and to E the mid point of AC. If the tension in this cord is 2 lbs. wt., find the force with which the rod AB acts on BC at B, and on AC at A. (First consider the equilibrium of the rod BC.)

## G.

**1.** A uniform straight bar AB, 3 ft. long, is bent at right angles at C, so that AC is twice BC. The end A rests on a smooth peg, and the end B is supported by a vertical string, so that AC is horizontal. Find the pressure on the peg and the tension in the string if the whole bar weighs 6 lbs.

**2.** A uniform beam AB, 12 ft. long, weighing 120 lbs., rests with its lower end B on the ground at the foot of a vertical wall BC. The other end A is attached by a rope AD to a point in the wall, 15 ft. vertically above B. If the beam is inclined to the wall at an angle of 30°, find the pressures of the beam against the wall and against the ground at B.

**3.** To the ends of a string 3 ft. long are fastened two equal spheres, each of radius 1 ft. and weight 5 lbs. The string is then hung symmetrically over a peg. Find the pressure between the spheres.

**4.** Find the resultant of the three given forces acting on a rigid body as in the diagram. *(Army.)*

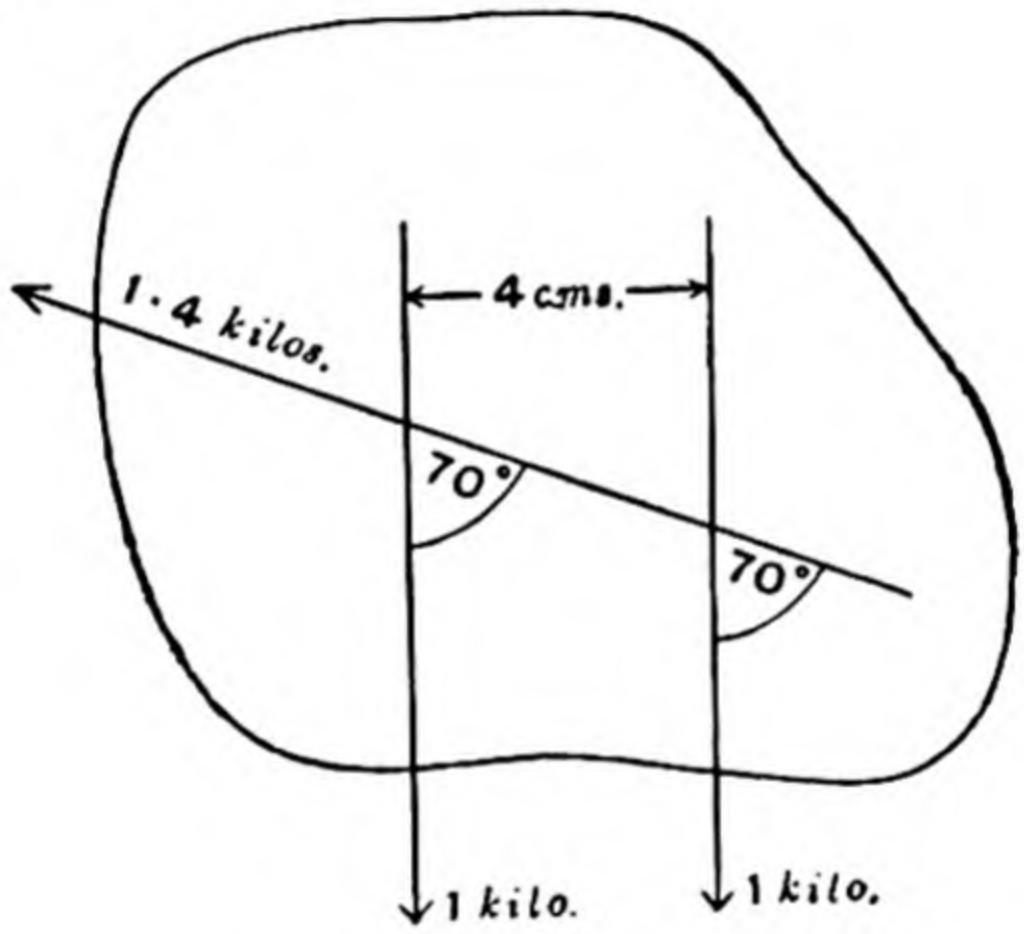



FIG. 99.

**5.** A uniform rod, of length $2a$, rests over the rim of a smooth hemispherical bowl fixed with its rim horizontal, one end being in contact with the inner surface of the bowl. If the rod makes an angle $\theta$ with the horizontal when in equilibrium, prove that $2r \cos 2\theta = a \cos \theta$, where $r$ is the radius of the bowl.

# CHAPTER IV.

## FRICTION.

*Experiment* 1. Take a well-planed block of wood, with a small staple fixed in the centre of one face, attach the dynamometer to the staple, and pull the block along a level table.

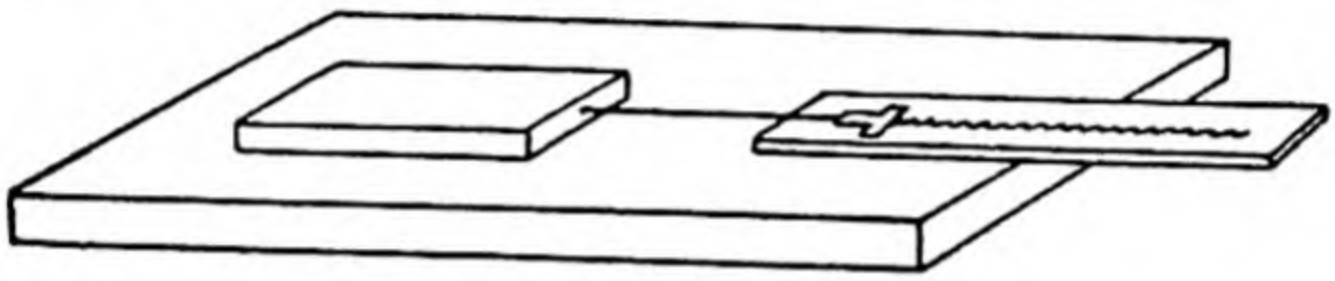


FIG. 100.

When the dynamometer pull is 10 grams wt., the block probably will not move.

What force is preventing its motion ? What is its magnitude ? In what direction is it acting ?

Increase the pull to 20 grams wt. ; if the block does not move, what is now the force of friction ?

It will be found that the force of friction varies so as to be just sufficient to prevent motion up to a certain maximum. When the pulling force is greater than this, the maximum frictional force is insufficient to prevent motion.

This limit which friction cannot exceed is called the *limiting force of friction.*

Take care to adjust the dynamometer for reading in a horizontal position.

Instead of noting the force required to start the block, find what force is required just to keep the block moving

with uniform velocity when it is started by the hand. It will be found that this force is less than was obtained before.

More uniform results are obtained by experimenting with surfaces in motion, and it is important to repeat an experiment under exactly the same conditions, using the same part of the table, and moving with the grain of the wood.

*Experiment* 2. To determine whether the maximum force of friction depends on the area of the surfaces in contact.

Fairly satisfactory results may be obtained by experimenting with the different faces of a block of wood, but these faces may differ in flatness, or in the state of the grain, and we require that the only variation shall be in the area in contact.

A block of glass, or metal, sliding on a paper surface, will be found most satisfactory.

*Experiment* 3. To determine whether the maximum force of friction depends upon the pressure between the surfaces in contact.

We can alter the pressure between the block and the table by placing weights on the block; the pressure between the surfaces will then equal the weight of the block added to the additional weight on the block.

The following table gives the results of an experiment performed with wood on wood:

| Pressure N. | Maximum Friction F. | Value of $\frac{F}{N}$. |
|---|---|---|
| 83 grams. | 20 grams. | ·24 |
| 133 ,, | 30 ,, | ·23 |
| 173 ,, | 38 ,, | ·22 |
| 233 ,, | 52 ,, | ·22 |
| 273 ,, | 58 ,, | ·21 |
| 333 ,, | 74 ,, | ·21 |
| 383 ,, | 85 ,, | ·22 |

If we plot N and F, we find that the points lie approximately on a straight line, which probably, as in this case, does not quite go through the origin.

It will be seen, however, that if we draw a straight line through the origin having as many points on one side as on the other, we get a very fair approximation to the truth, so that for practical purposes it is sufficiently accurate to assume that $\frac{F}{N}$ is a constant for the given surfaces.

In this case we may take $\frac{F}{N} = 0{\cdot}22$.

If we experiment with glass on wood, we shall again find $\frac{F}{N}$ to be approximately a constant, though not the same constant as for wood on wood.

The fraction $\frac{F}{N}$ is usually called $\mu$, and the value of $\mu$ is termed the *coefficient of friction*.

*Hence* *Limiting Friction* $= \mu N$.

The value of $\mu$ depends upon the nature of the substances in contact and the character of their surfaces; it will only be constant for two given specimens; and values obtained, say, for one piece of wood will not necessarily be the same for another specimen of the same kind of wood.

The Laws of Friction may be thus stated:

(1) Friction acts so as to oppose relative motion.

(2) The magnitude of the force will vary so as just to prevent motion up to a certain maximum value.

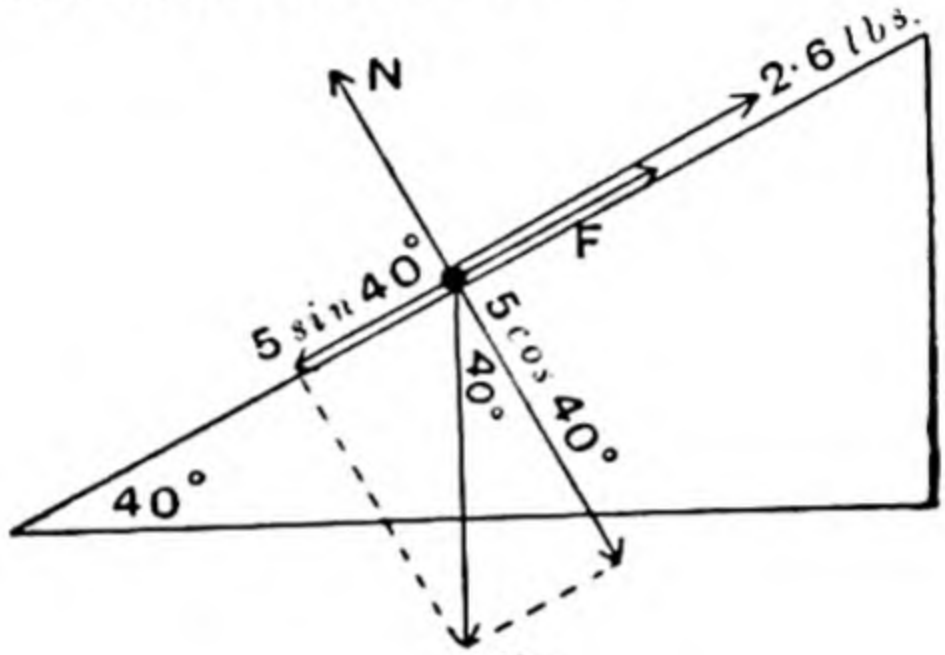
(3) The force is independent of the areas in contact.

(4) The maximum force of friction for two given surfaces is $\mu N$, where N is the normal pressure between the surfaces, *i.e.* the pressure at right angles to them.

*N.B.*—Laws 3 and 4 are only approximately true. All

our experiments have been performed with dry surfaces; the laws are quite different if lubricants are used, and the results also depend upon the velocity of the bodies.

**Example.** A body weighing 5 lbs. is just prevented from slipping down a rough plane inclined at 40° to the horizontal by a force of 2·6 lbs. acting up the plane; find the least force up the plane which would be required to start the body moving up, and find the coefficient of friction.




5 *lbs.*

Fig. 101.

Resolve the weight, 5 lbs., into two components, $5\cos 40°$ at right angles to the plane and $5\sin 40°$ down the plane.

Let F be the maximum force of friction.

Since the body is on the point of slipping down the plane, F will act up the plane.

$$\therefore\ 2{\cdot}6 + F = 5\ \sin 40°;$$

$$\therefore\ F = 5\ \sin 40 - 2\ 6 = {\cdot}6.$$

When the body is on the point of moving up the plane, F will act down the plane.

Therefore, if P be the force required to move the body up the plane,

$$P = 5\sin 40 + F = 10\sin 40 - 2{\cdot}6 = 3{\cdot}8 \text{ lbs.}$$

The normal pressure on the plane is $5\cos 40°$.

$$\therefore\ \mu = \frac{F}{N} = \frac{{\cdot}6}{3{\cdot}8} = {\cdot}16.$$

## EXAMPLES XVII.

**1.** A book is lying on the tabie. If the table were moved horizontally, what force causes the book to move? If there were no friction between the book and the table, what would happen?

**2.** Find what force would be required to keep a body weighing 50 lbs. moving along a horizontal surface if the coefficient of sliding friction were 0·2.

**3.** A horizontal force of 10 lbs. wt. will just move a body weighing 40 lbs. along a horizontal plane; find the coefficient of friction.

**4.** A horizontal force of 2 lbs. wt. will just move a board weighing 8 lbs. along a horizontal plane. What force would be required, if a weight of 12 lbs. were placed on the board?

**5.** A weight of 20 lbs. rests on a plane inclined at 25° to the horizontal. What is the force of friction supporting it?

**6.** A body weighing 50 lbs. rests on a plane inclined at 20° to the horizontal. What is the normal pressure on the plane? If the body is on the point of slipping, find the coefficient of friction.

**7.** A load weighing 30 lbs. rests on a horizontal plane, and is pulled by a force of 10 lbs. wt. inclined at 20° to the horizontal. What is the normal pressure on the plane? If the body does not move, what is the frictional force?

**8.** A block of stone weighing 1 cwt. is pulled horizontally by a force of 55 lbs. wt. What is the tangent of the angle the resultant of friction and the normal pressure makes with the normal to the surface? If the block just begins to move, what is the limiting friction? What is the coefficient of friction?

**9.** A weight of 20 lbs. is on the point of slipping down a plane inclined at 3C° to the horizontal. In what direction does the resultant of the normal pressure and the friction act? What is the angle between this resultant and the normal?

**10.** A body weighing 200 grams rests on a horizontal plane, and is pulled by a force of 50 grams inclined at 40° to the

horizontal. Find the difference between the normal pressure on the plane and the weight of the body. If the body is on the point of slipping, find the force of friction and the coefficient of friction.

**11.** A body weighing 10 lbs. rests on a rough plane inclined at 20° to the horizontal. A force of 1 lb. down the slope of the plane will just make the body slide down with uniform velocity. Find the frictional force. What is the least force in the opposite direction that would keep the body moving uniformly up the plane? Find the coefficient of friction.

**12.** A body resting on a horizontal plane is pulled by two ropes at right angles to one another, acting parallel to the plane. The tensions of the ropes are 20 lbs. and 15 lbs. respectively. If the block does not move, find the force of friction that is being exerted.

**13.** A body weighing 20 lbs. is placed on a plane inclined at 20° to the horizontal, the coefficient of friction being ·4. Find the least force, parallel to the plane, necessary to start the body moving up the plane and also to start it moving down the plane.

**14.** A block weighing 40 lbs. can just be moved on a horizontal plane by a force of 12 lbs. acting at 20° to the plane. What force acting at 30° to the plane would be required?

**15.** A box weighing 5 cwt. is slowly dragged up a plane inclined to the horizontal at 35°. If $\mu = 0{\cdot}4$, find the work done in moving the box 10 ft. up the plane.

**Angle of Friction.**

It is frequently convenient to combine the tangential force of friction and the normal reaction due to the surface into a single force.

When friction is a maximum, we may take as an approximation $F = \mu N$.

If OB is the line of action of this resultant R, and $\lambda$ the angle it makes with the normal OA, we have

$$\tan \lambda = \frac{AB}{OA} = \frac{\mu N}{N} = \mu.$$

This angle λ between the common normal to the surfaces, and the resultant R of the limiting friction and the normal

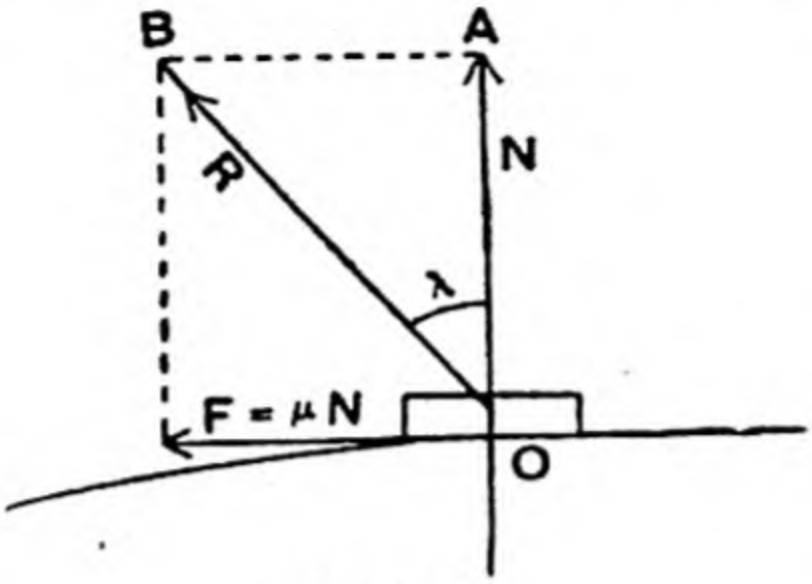



FIG. 102.

pressure is called the *angle of friction*. We note that the tangent of this angle $= \mu$.

If friction is not at its maximum value, it will be less than μN, so that the angle between the normal and the direction of the total reaction R is less than λ.

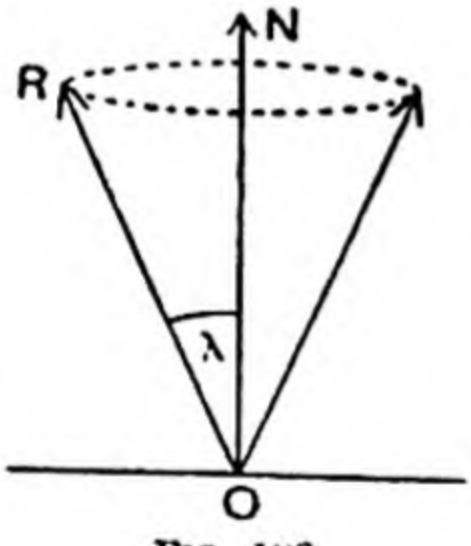



FIG. 103.

If the body in contact with a rough surface at O is pulled in any direction, the frictional force will always act so as to oppose the motion, and the maximum value of the angle between the normal and the resultant reaction will always be λ, so that the line of action of this resultant will describe a cone with the normal as axis, and semi-vertical angle $= \lambda$. This cone is called the cone of friction.

*Experiment.* CD is a brass stair-rod passing through two corks C and D, so as to be firmly held by the clamps of the retort stand. AB is a light rod with brass eye screws at A and B, so that the eye at A slides freely on CD.

Fasten a weight of 500 grams at B, and connect B to C by a string.

Adjust AB until A is on the point of slipping up the rod DC. If AB is light, we may neglect its weight in comparison with the other forces acting on it.

It is therefore in equilibrium under the resultant reaction at A, and the resultant of T and the 500 grams wt. at B.

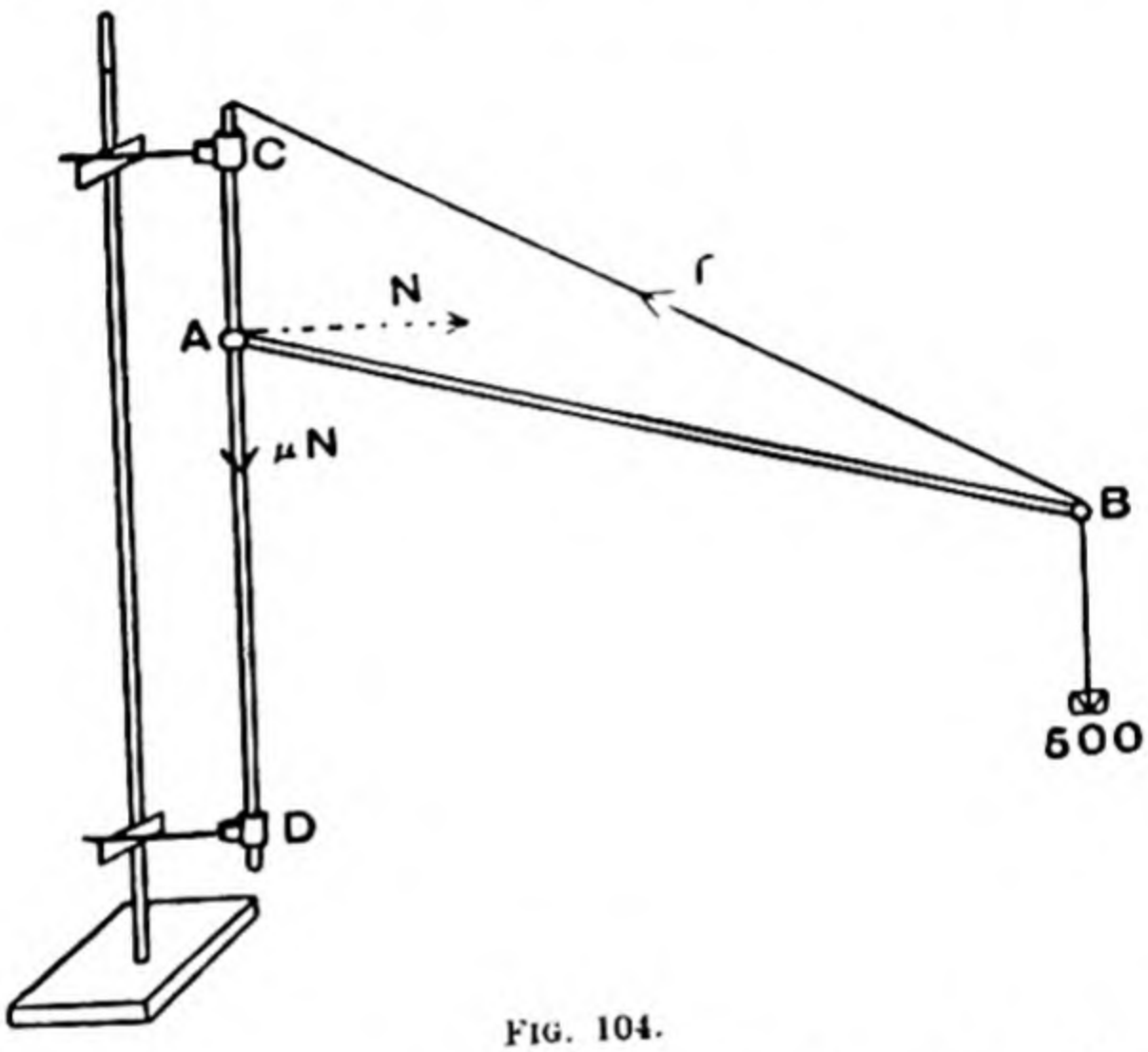



FIG. 104.

These two forces must act along the line AB. Therefore the resultant reaction at A is along AB. We can now measure the angle of friction for brass on brass.

Another position may be found where A is on the point of slipping down CD.

**Graphical Solutions.**

Problems including frictional forces can often be solved graphically by taking the resultant of the maximum friction and the normal reaction acting at an angle $\lambda$ to the normal.

EXAMPLE. A body rests on a rough horizontal plane, and is pulled by a force making an angle $\theta$ with the plane. If the coefficient of friction is $\mu$, find for what value of $\theta$ the force will be a minimum.

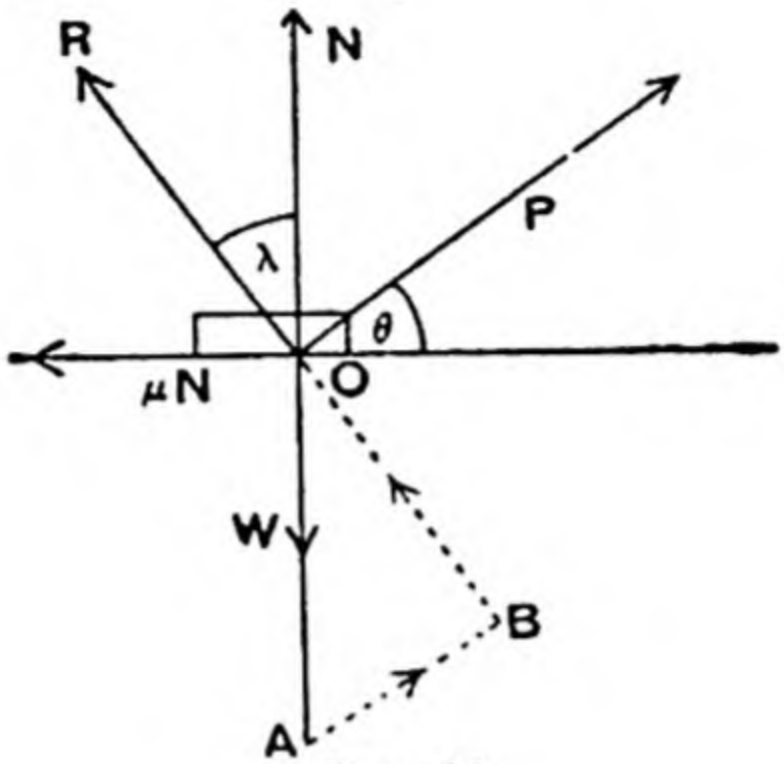



FIG. 105.

Since friction is a maximum, the resultant of N and F will act at an angle $\lambda$ to the normal, where $\tan\lambda = \mu$.

The body is now acted upon by three forces, W, R and P.

If we draw a triangle of forces OAB, where OA represents W, it is clear that AB, representing P. will be a minimum when AB is perpendicular to OB, *i.e.* when P acts at right angles to OR. .. for P to be a minimum, $\theta = \lambda = \tan^{-1}\mu$.

If a body is resting on an inclined plane supported only by friction, when it is on the point of slipping $F = \mu N$, and the resultant of F and N must make an angle $\lambda$ with the normal. (Fig. 106.)

Why must the direction of this resultant be vertical?

Show, by geometry, that $\lambda = a$.

We can therefore find the coefficient of sliding friction between a block of wood and a board, by finding at what angle the board must be inclined to the horizontal before the block will continue to slip downwards when gently started.

If the coefficient of sliding friction is $\mu$, it is important to notice that, if the plane were tilted to a greater angle,

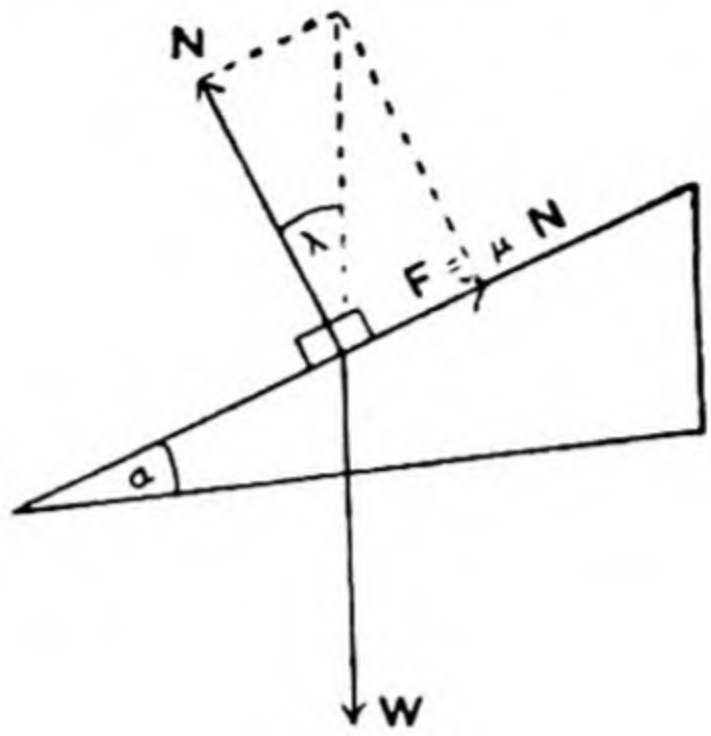



FIG. 106.

the force of friction during the motion would depend upon $\alpha$, since the normal reaction will be $W \cos \alpha$, and therefore diminishes as $\alpha$ increases.

## EXAMPLES XVIII.

**1.** A block of wood weighing 50 lbs. just begins to slip down an inclined plane when it is tilted at an angle of 40° to the horizontal. Find the coefficient of friction.

**2.** A board is placed on the surface of a sphere of radius 1 ft., and it is found to slip when it touches at a point 1 in. from the highest point of the sphere, measured vertically. Find the angle of friction.

**3.** A light stick has a ring at one end A, which slides on a vertical brass rod. At the other end B hangs a weight of 300 grams, and B is fastened to C, a point in the brass rod above A. If the ring slips when the stick makes an angle of 75° with the vertical, find the coefficient of friction between the ring and the brass rod

4. A body weighing 200 grams is placed on a board which is tilted until the body begins to slide. If the coefficient of statical friction is 0·3, find at what angle sliding will take place.

5. A body weighing 1 cwt. is just moved along a horizontal plane by a force acting at an angle to the horizontal. If the angle of friction between the body and the plane is 15°, find by drawing, the direction of the force, so that it may be the least possible. What would then be its magnitude ?

6. A cubical block of wood, 4 ft. each way, rests on a rough horizontal plane, and is on the point of moving when pulled by a horizontal force at right angles to one face, acting at the mid point of the top edge of that face. If the angle of friction is 10°, find at what point in the base the resultant reaction of the ground will act. What must be the least angle of friction if the block will tilt before slipping ?

7. A body weighing 1 cwt., placed on an inclined plane whose angle of inclination is 30°, is acted upon by a force P, which is inclined to the horizontal at an angle of $\theta$, so that it just pulls the body up the plane. If the angle of friction is $\lambda$, find, by drawing a triangle of forces, the value of $\theta$ for which P will be a minimum. If $\lambda = 20°$, find P.

8. A uniform plank, 20 ft. long, weighing 120 lbs., is placed with its mid point resting on a rough horizontal cylinder of radius 2 ft., so that the plank lies at right angles to the axis of the cylinder. If the angle of friction is 30°, find through what angle the plank may be tilted before it begins to slip when a weight W is placed at the end of the plank. Find W.

9. One end of a ladder rests on the rough ground, and the other end against a smooth vertical wall. The C. of G. of the ladder is one-third of its length from the bottom, and the ladder begins to slip when it makes an angle of 40° with the ground. Find graphically the coefficient of friction between the ladder and the ground.

10. A uniform rod AB, 4 ft. long, rests with the end B on a rough horizontal plane, the end A being supported by a string tied to a point 5 ft. vertically above B. If the rod is on

the point of slipping when it makes an angle of 20° with the vertical, find the coefficient of friction at B.

**11.** A ring weighing 5 lbs. hangs on a rough horizontal rod. A string is fastened to the top of the ring, and when inclined upwards at 30° to the rod, it just moves the ring along the rod if it is pulled with a force of 1 lb. wt. Assuming the plane of the ring remains vertical, find the coefficient of friction, and find what force would be required to move the ring if the string were parallel to the rod.

**12.** A motor is driven with uniform velocity up a hill whose slope is 1 in 10. If the axle of the driving wheels now supports 0·6 of the weight of the car, find the least coefficient of friction between the tyres and the ground.

**13.** The total weight of a train is 250 tons, and the resistances to its motion are equivalent to 10 lbs. wt. per ton. If the load taken by the driving wheels is 30 tons, find the least value for $\mu$ between the driving wheels and the rails if the wheels are not to skid when the engine starts working.

**14.** Two spheres, whose weights are W and W′, rest near the top of different and differently inclined planes placed back to back. The highest point of the spheres are connected by a horizontal string perpendicular to the common horizontal edge of the two planes and above it. If $\mu$, $\mu'$ be the coefficients of friction, and be such that each sphere is on the point of slipping down, prove that $\mu W = \mu' W'$.

**15.** The diagram represents a grab for lifting stone blocks. If $A\hat{C}B = 120°$, find the least coefficient of friction at F and E in order that the grab may lift a block of stone, 5 feet cube, weighing 4 tons. Neglect the weight of the grab.

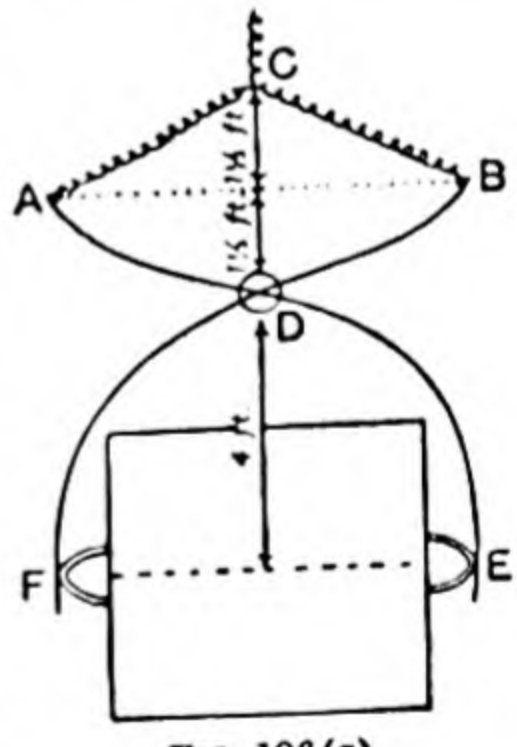


FIG. 106(a).

## Laboratory Experiments.

**1.** Experiments in the text (pp. 113–115).

**2.** Fasten 100 grams wt. to a dynamometer by a string, and let the string hang over a glass tube about 1 cm. in diameter fixed horizontally by a clamp.

Hold the dynamometer horizontally at right angles to the tube so that the two parts of the string are at right angles to one another. Adjust the dynamometer for reading in this position, and note the reading when the 100 gram wt. is slowly drawn upwards and also when it is just slipping down. Note the force of friction in each case

Repeat when the angle between one part of the string and the other part produced (*i.e.* the angle through which the string is bent) is 45°, 90°, 135°, 180°, 225°, etc., approximately. Plot the relation between the friction and the angle on squared paper.

State whether the force needed to pull up the weight is proportional to the amount of string in contact with the tube.

Repeat with a much wider tube, and compare your results with those in the previous experiment.

**3.** From the smooth curve drawn to show the relation between the angle and the force required to raise the weight in the first experiment with the tube, take the corrected values of the force for intervals of 45°.

Look up the logarithms of these numbers, and plot a graph showing the relation between the logarithm of the force needed to raise the weight and the angle through which the string is bent.

A smooth curve through these points should be a straight line, showing that the logarithm of the force is proportional to the angle.

# CHAPTER V.

## CENTRE OF GRAVITY.

If a weight $W_1$ acts on a light rod at a point A, and another weight $W_2$ at a point B, we have seen (p. 26) that these weights will balance if their moments about the fulcrum C are equal and tend to turn the rod in opposite directions. The rod will also be in equilibrium if the fulcrum is supported by a vertical force equal to the sum of the weights.

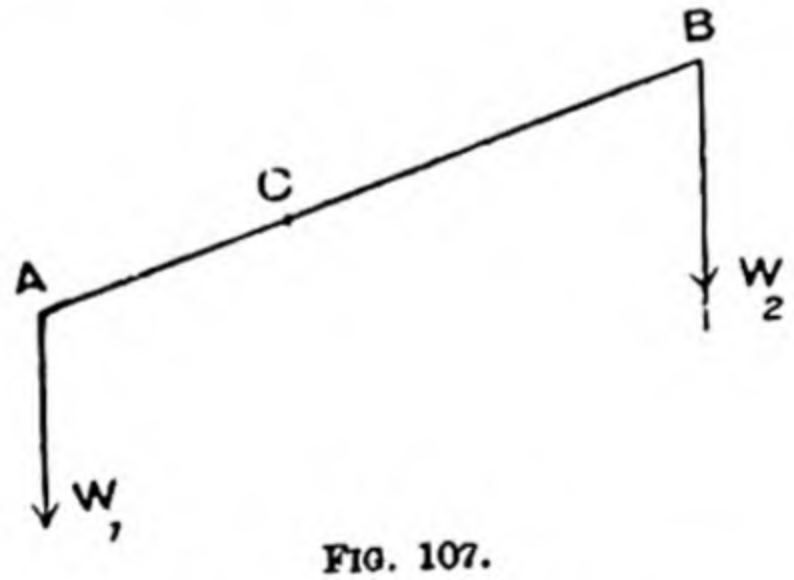



FIG. 107.

If the rod is turned through an angle round the fulcrum, the weights will still balance, for their moments about C will still be equal.

If we attach another pair of weights $W_3$, $W_4$ to the rod at $A_1$ and $B_1$, these will also balance about C if

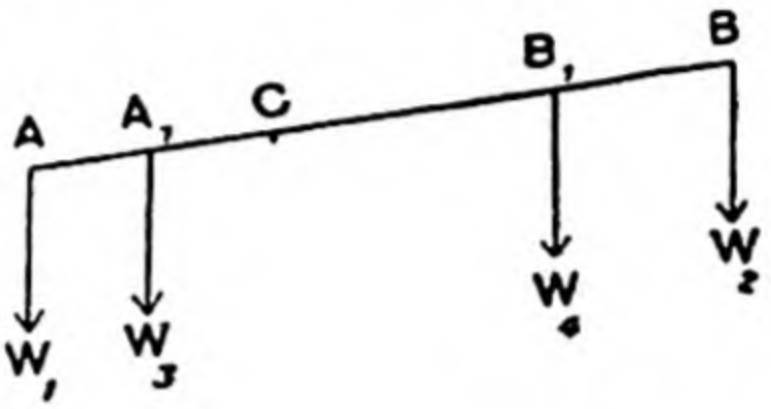
$$W_3(A_1C) = W_4(B_1C).$$

This pair of weights could also be placed on a rod $A_1B_1$, intersecting AB at C, and would balance at C if

$$W_3(A_1C) = W_4(B_1C).$$

even when these weights do not lie in the same vertical plane as $W_1$ and $W_2$.

The whole system will now balance about C, and could be supported by a force $W_1 + W_2 + W_3 + W_4$ at C.



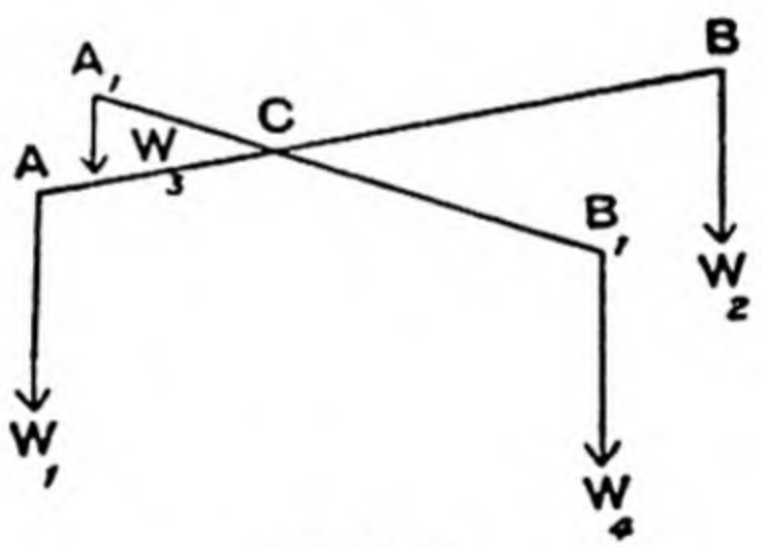




FIG. 108.

Any rigid body may be considered as made up of a system of particles rigidly joined together, and each possessing its own weight.

If we can find a point about which this system will balance in any position, it could be supported by a force at that point, equal to the total weight of the particles.

The point about which a body will balance in all positions is called the *Centre of Gravity* of the body, and the resultant

of the weights of all the particles composing the body always passes through this point.

If, then, we can divide a body up into pairs of particles, and find by inspection a point about which each pair will balance, that point will be the Centre of Gravity of the body.

**Centre of Gravity of a Straight Line or Uniform Thin Rod.**

This will obviously be the mid point of the rod, since the rod may be divided into pairs of equal particles equidistant from this point.

**Centre of Gravity of a Parallelogram.**

The parallelogram may be divided into thin strips (A′B′) parallel to AB.

Prove that the line EF, joining the mid points of AB and DC, will bisect each strip.

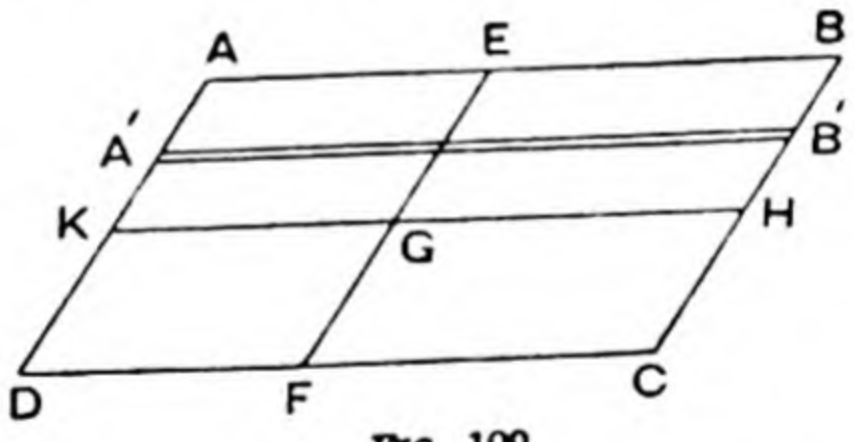



FIG. 109

The C. of G. of the figure, therefore, lies in EF.

Similarly, by dividing the figure into strips parallel to AD, we see that the C. of G. lies in KH, which joins the mid points of AD and BC. Therefore the C. of G. is at G, where KH and EF intersect.

This point is also the point of intersection of the diagonals.

*Experiment.* Cut out from stiff cardboard a triangle ABC.

Suspend the triangle by a pin close to A, so that the triangle can turn freely in a vertical plane.

Mark on the cardboard the vertical line which passes through the pin. (Find this direction by means of a plumb line.) Now suspend the cardboard at B, and mark the vertical line through B.

Let these two lines intersect at G.

Suspend the cardboard at C, and verify that the vertical through C passes through G.

Test whether the cardboard will balance in a horizontal position when supported on a vertical pencil whose point is placed at G.

Test by measurement the following:

The lines drawn on the triangle are the medians.

The point G divides each median in the ratio 2 : 1.

**Centre of Gravity of a Triangular Lamina.**

(A lamina is a thin sheet whose thickness may be neglected.)

*Theoretical proof.* Divide the triangle into thin strips, such as *bdc* parallel to BC.

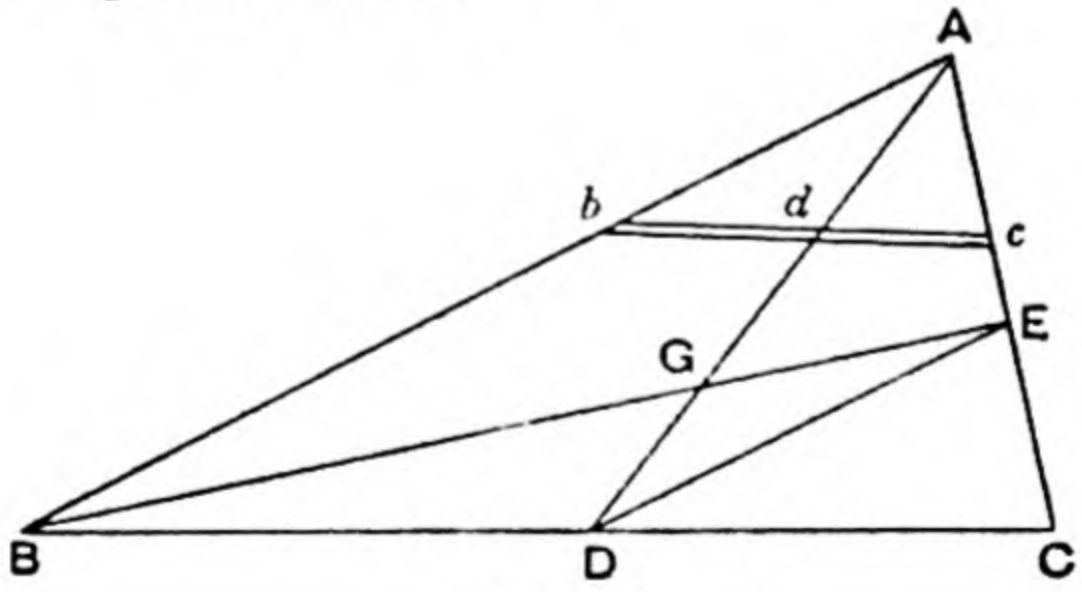



FIG. 110.

Join A to the mid point of BC, then AD bisects each strip, such as *bdc*, and the C. of G. of the lamina lies in AD.

Prove this by the similar triangles, A*db* and ADB, A*dc* and ADC.

Similarly, dividing the lamina into strips parallel to AC, we see that the C. of G. lies in BE. Therefore it is at G, where the medians intersect.

Join DE.

From similar triangles ABG, GDE we have

$$\frac{AG}{GD} = \frac{AB}{DE} = \frac{2}{1};$$

$$\therefore\ AG = 2GD;$$

$\therefore$ G is one-third of AD from D.

**Centre of Gravity of three Equal Weights placed at the Corners of a Triangle.**

The resultant of W at B and W at C will be 2W at D, the mid point of BC. The resultant of 2W at D and W at A will

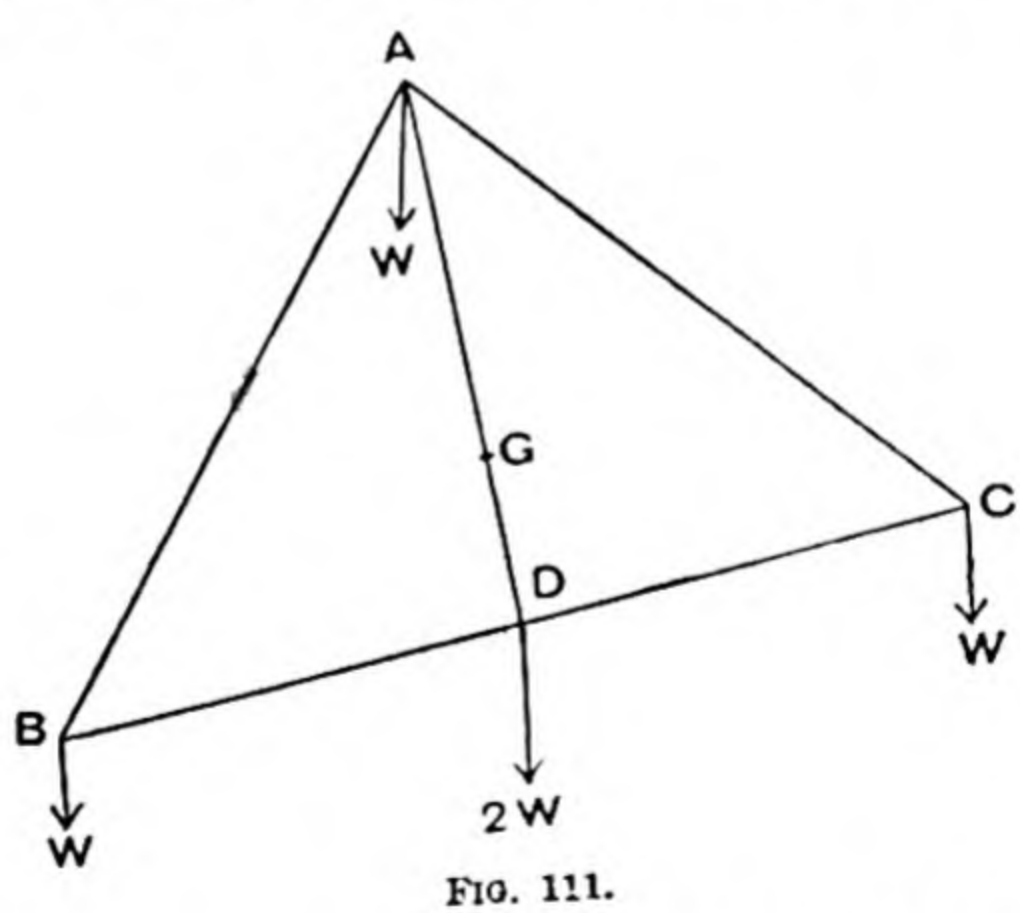



FIG. 111.

act at G on AD, such that AG : GD = 2 : 1. This is the same point as the C. of G. of the triangular lamina ABC.

**Centre of Gravity of a Quadrilateral.**

Divide the quadrilateral into two triangles by drawing the diagonal AC.

Let the C. of G. of ABC be at $G_1$ on the median BE, and the C. of G. of ADC at $G_2$. Then the C. of G. of the whole quadrilateral lies or $G_1G_2$.

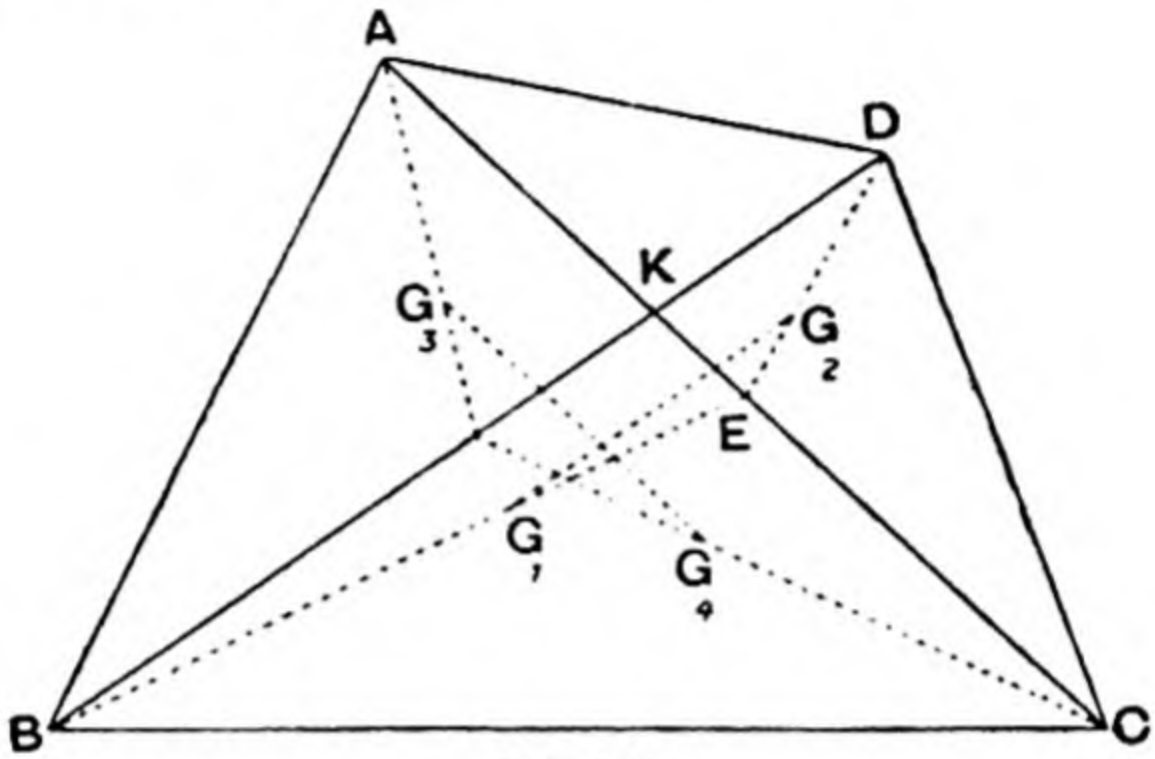



FIG. 112.

Draw the diagonal BD dividing the quadrilateral into two triangles whose C. of G. are at $G_3$ and $G_4$.

Then the C. of G. of the whole quadrilateral is at G, where $G_1G_2$, $G_3G_4$ intersect.

*Alternative Method.*

Let the diagonals AC, BD intersect at K. From CA mark off CL = AK.

The C. of G. of the quadrilateral coincides with the C. of G. of the triangle DBL.

*Proof.* Show that $G_2$ is the C. of G. of the triangles DKL and DAC, also that $G_1$ is the C. of G. of the triangles BKL and BAC.

Then the C. of G. of the two triangles DAC and BAC will divide $G_2G_1$ in the ratio BAC : DAC, and the C. of G. of the triangles DKL, BKL will divide $G_2G_1$ in the ratio BKL : DKL.

Now $\frac{BAC}{BKL} = \frac{AC}{KL}$ and $\frac{DAC}{DKL} = \frac{AC}{KL}$.

∴ the C. of G. of the quadrilateral ABCD coincides with the C. of G. of the triangle DBL, and may be found by drawing two medians of the triangle DBL.

## EXAMPLES XIX.

**1.** The sides of a triangular lamina are 3 ft., 4 ft. and 5 ft. Find the distance of the C. of G. from the right angle.

**2.** A triangular lamina ABC is suspended freely at A. Prove that the vertical line through A will pass through the mid point of BC.

**3.** A right-angled triangular lamina ABC is freely suspended at A. Find the angle AB makes with the vertical if AB = 9 in., BC = 12 in., CA = 15 in.

**4.** A triangular board is suspended by three vertical strings at its angular points. If the board is horizontal, prove that the tensions in the strings will be equal to one another.

**5.** The figure is composed of three rectangular strips of equal length and breadth. Draw the figure and find its C. of G. If K, L, M be the mid points of the three rectangles, prove that the C. of G. of the figure coincides with the C. of G. of the △KLM.

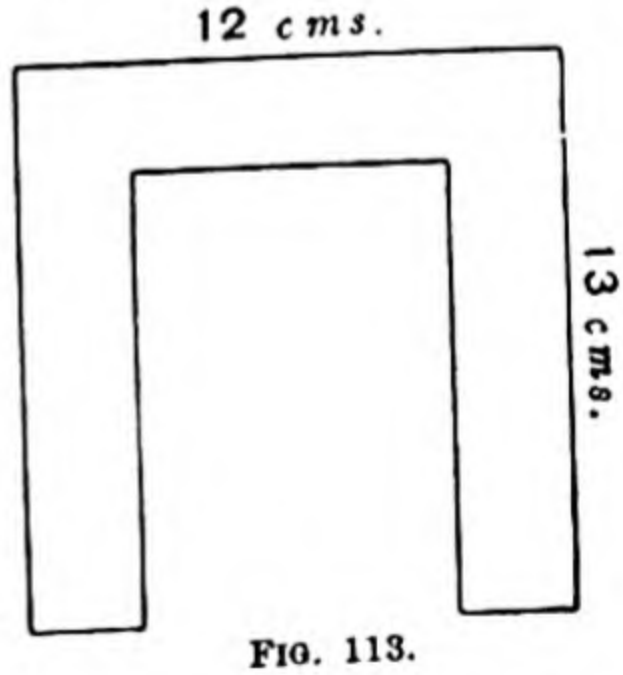


FIG. 113.

**6.** Find graphically the distance from the parallel sides AB, DC of the C. of G. of a trapezium ABCD, given AB = 8 cms., DC = 10 cms., AD = 5 cms., AD being perpendicular to AB and DC.

**7.** Three men carry a triangular board by supporting it on their shoulders, placed at the angular points. Where must a weight of 10 lbs. be placed on the board, so that the pressure on each man may be the same?

**8.** Two isosceles triangles, whose heights are 12 in. and 9 in. respectively, have the same base, but are on opposite sides of it in the same plane. Find the position of the C. of G. of the whole figure.

**9.** Plot out on squared paper the points A(−2, 4), B(2, 5), C(3, 2), D(0, −1). Find graphically the C. of G. of the figure ABCD.

**10.** A uniform quadrilateral ABCD lies on a horizontal plane. If the corner A is lifted, state what is the condition which determines whether the side BC or the side CD will leave the plane first.

**Moment of the Resultant.**

The principle of the lever has shown that when a weight $w_1$ acting at A balances a weight $w_2$ acting at B, the equili-

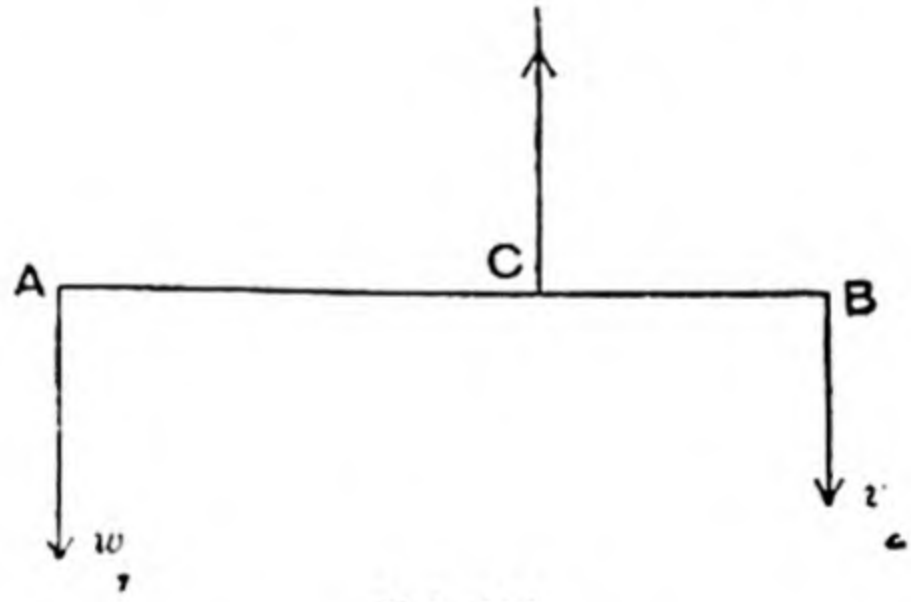



FIG. 114.

brant $w_1 + w_2$ acts at C in the line AB where $w_1(AC) = w_2(BC)$, *i.e.* the resultant of $w_1$ and $w_2$ will act at C, and it is parallel to the line of action of $w_1$ and $w_2$.

Let O be any point in the plane of $w_1$ and $w_2$.

Drop a perpendicular from O on the lines of action of $w_1$ and $w_2$, cutting them at A and B, and cutting the line of action of their resultant in C; then $w_1(AC) = w_2(BC)$.

Now, the moment of $w_1$ about O + moment of $w_2$ about O

$$= w_1(OA) + w_2(OB)$$
$$= w_1(OC - AC) + w_2(OC + CB)$$
$$= w_1(OC) - w_1(AC) + w_2(OC) + w_2(CB)$$
$$= (w_1 + w_2)OC$$
$$= \text{moment of the resultant of } w_1 \text{ and } w_2 \text{ about O}$$

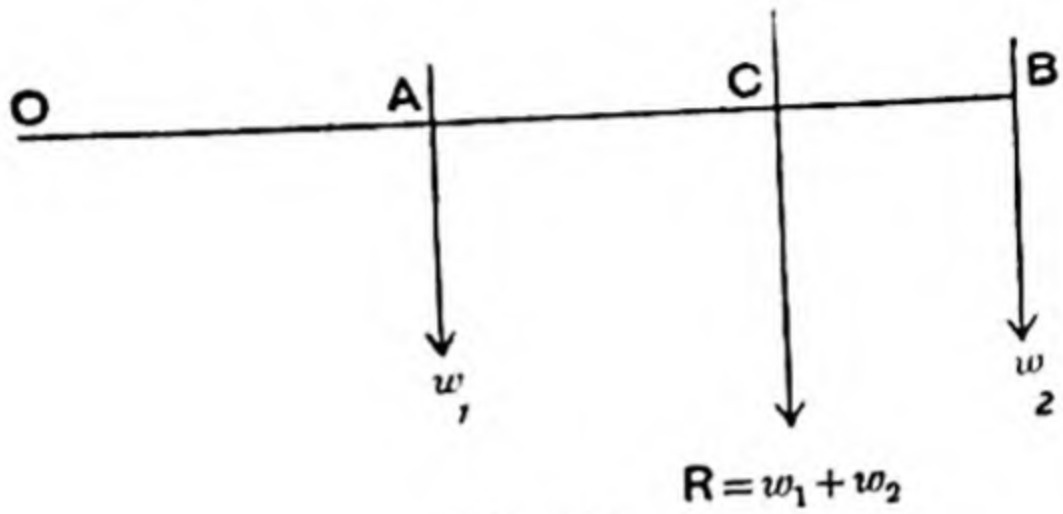



FIG. 115.

If O lies between the lines of action of $w_1$ and $w_2$, we must take the difference of the moments of $w_1$ and $w_2$ about O to get the moment of their resultant about O, since $w_1$ and $w_2$ will tend to turn a lever round O in opposite directions.

If we call a moment in one direction $+^{ve}$, and a moment producing a rotation in the other direction $-^{ve}$, we may say that in both cases the Algebraical Sum of the moments of two weights about any point in their plane equals the moment of their resultant about that point.

This result may be extended to apply to any number of such forces.

If there is a third weight $w_3$, then the sum of the moments $w_3$ and the resultant of $w_1 + w_2$ about O equals the moment of the resultant $w_1 + w_2 + w_3$ about O; *i.e.* the sum of the moments of $w_1$, $w_2$ and $w_3$ about O equals the moment of their resultant $w_1 + w_2 + w_3$ about O.

**To find the Centre of Gravity of a System of Weights acting at Points in a Plane.**

Let a system of weights $w_1, w_2, \ldots$ act at points in a plane whose coordinates referred to two rectangular axes on the plane are $x_1y_1$, $x_2y_2$, etc.

Let the coordinates of the C. of G. be $\bar{x}$, $\bar{y}$.

Place the plane in a vertical position, so that OX is horizontal.

Take moments about O. The sum of the moments of $w_1$, $w_2$, etc., about O equals the moment of their resultant about O;

$$\therefore\ (w_1+w_2+w_3+\ldots)\bar{x}=w_1x_1+w_2x_2+\ldots;$$

$$\therefore\ \bar{x}=\frac{w_1x_1+w_2x_2+\ldots}{w_1+w_2+w_3}=\frac{\Sigma wx}{\Sigma w}.$$

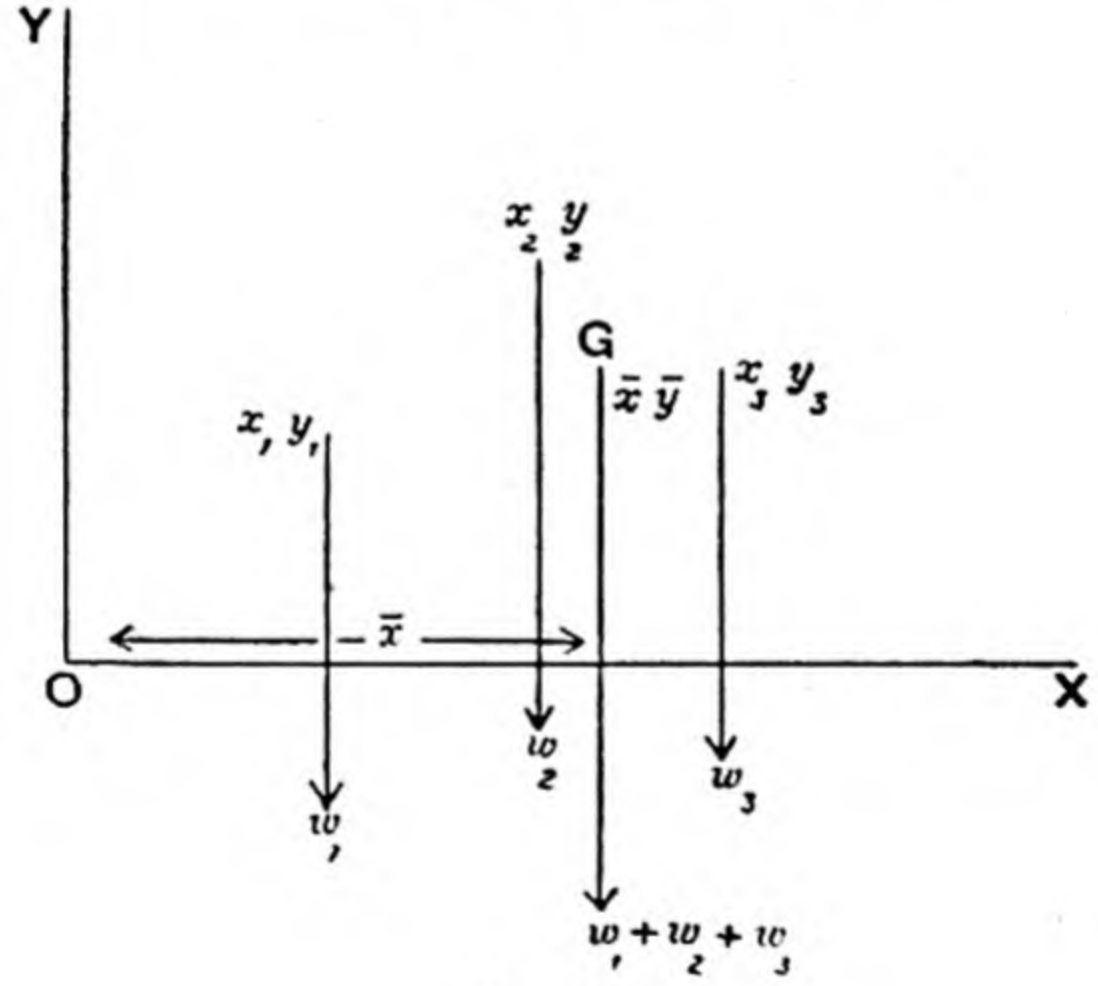



FIG. 116.

$\Sigma wx$ is read "Summation $wx$," and means the sum of all terms of the form $wx$.

Imagine the plane turned through a right angle round O, so that OY is now horizontal (Fig. 117).

Again, taking moments about O, we have

$$\bar{y} = \frac{\Sigma wy}{\Sigma w}.$$

If we imagine the plane to be horizontal (Fig. 118), we get the same values as before for $\bar{x}$ and $\bar{y}$ if we multiply

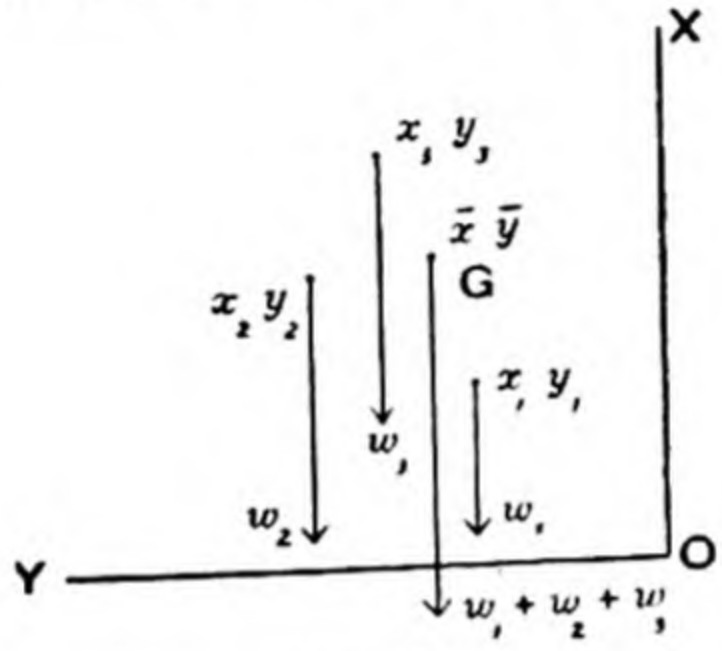



FIG. 117.

each weight by the distance of the corresponding point $(x_1, y_1)$ $(x_2, y_2)$ etc., from the lines OX, OY respectively. This is called taking moments about a line, so that taking moments about OY we get $\Sigma w \,.\, (\bar{x}) = \Sigma(wx)$, and taking moments about OX we have $\Sigma w \,.\, (\bar{y}) = \Sigma(wy)$.

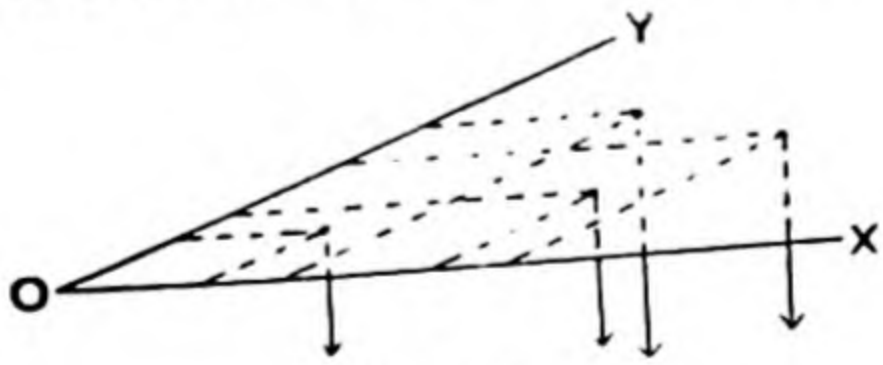



FIG. 118.

It is shown in text-books on Dynamics that the weight of a body is proportional to its mass, so that if we have a

series of masses $m_1$, $m_2$, etc., at points whose coordinates are $x_1y_1$, $x_2y_2$, the above results become

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\ \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

The point determined by these coordinates is called the Centre of Mass.

If the weights all act at points along a straight line, we may take this line as the X axis, and only one coordinate of the C. of G. will require to be determined.

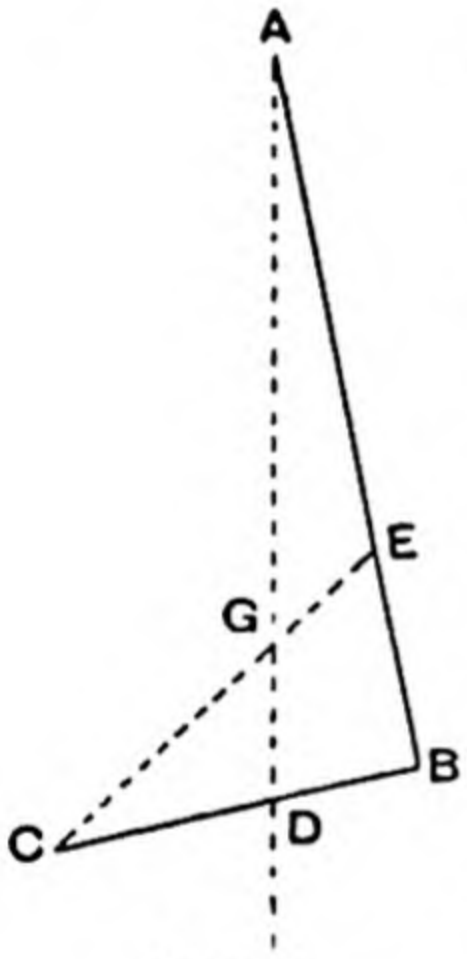


FIG. 119.

*Experiment.* Bend a piece of wire so as to form a right angle, with one arm AB twice the length of the other arm.

Hang the wire up by a string tied to A, and mark on BC the point D where the vertical through A cuts the part BC; then the C. of G. of the wire is in AD.

Now, hang the wire up at C and mark E, the point in AB where the vertical cuts it.

These two lines intersect at the C. of G. of the wire.

Taking BA and BC as axes, calculate the coordinates of

FIG. 120.

the C. of G. of the wire, and compare with the position obtained by experiment.

**Example 1.** ABC is an isosceles triangle, AB = AC = 3 ft., BC = 2 ft. Find the C. of G. of weights of 3 lbs., 4 lbs., 5 lbs. placed at A, B, C respectively.

The resultant will be 12 lbs. at G, whose coordinates referred to BC and BD are $\bar{x}$, $\bar{y}$.

Suppose DB to be vertical, and take moments about B.

Then $12\bar{x} = 4 \times 0 + 3 \times 1 + 5 \times 2$.

$$\therefore \bar{x} = 1{\cdot}1 \text{ ft.}$$

Suppose the figure rotated through a right angle so that BC is vertical, and again take moments about B. Then

$$12\bar{y} = 4 \times 0 + 5 \times 0 + 3 \times \sqrt{8}.$$

$$\therefore \bar{y} = {\cdot}71 \text{ ft.}$$

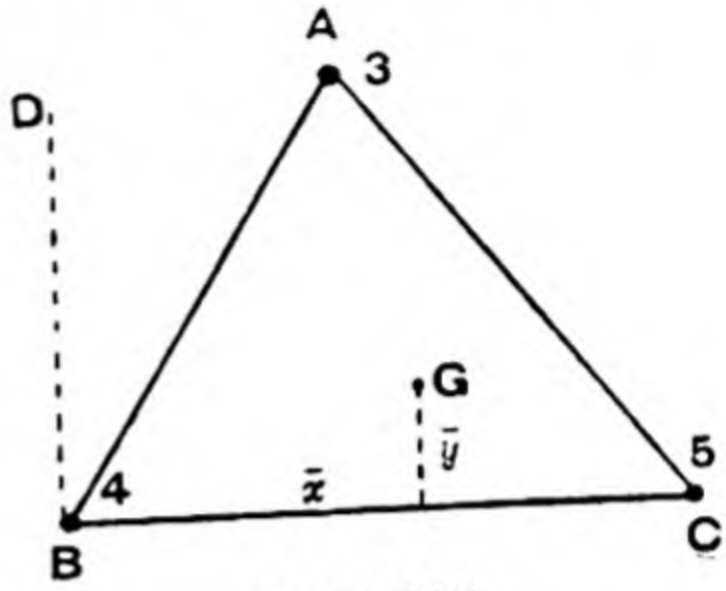



Fig. 120 (*a*).

These results are the same as if we supposed the plane of the triangle to be horizontal and took moments about the lines BD and BC respectively.

**Example 2.** A uniform circular table weighing 30 lbs. is supported by 3 legs placed symmetrically at the edge. If 12 lbs. is placed at D and 10 lbs. at E (see figure), find the pressures on the legs.

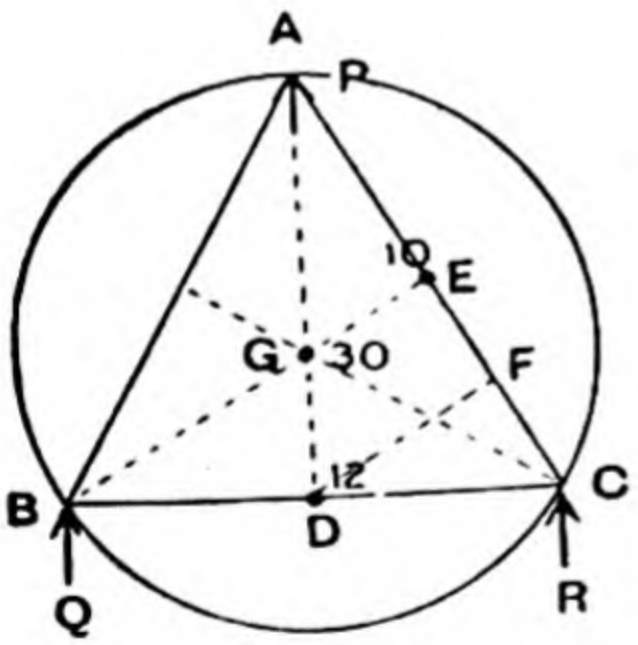



Fig. 120 (*b*).

Let P, Q, R be the pressures of the legs on the table.

Take moments about the line AC.

Let BE = $a$; then, since DF = $\frac{a}{2}$,

$$Q \times a = 30 \times \frac{a}{3} + 12 \times \frac{a}{2} + 10 \times 0;$$

$$\therefore Q = 16 \text{ lbs.}$$

Similarly, by taking moments about the line BC we may find P.

## EXAMPLES XX.

1. Weights of 1, 2, 4, 5 lbs. are spaced out along a line at equal distances of 1 ft. Find the position of their C. of G.

2. Weights of 2 lbs., 3 lbs., 5 lbs., 6 lbs. are placed at the corners A, B, C, D of a square, whose side is 1 ft. Find the distance of their C. of G. from the sides AB and AD.

3. A telescope consists of three tubes, each 10 in. in length, and of weights 10, 8, 6 ozs. Find the position of the C. of G. of the tubes when they are drawn out at full length.

4. A pendulum consists of a uniform bar 2 ft. long, weighing 4 ozs., and a spherical ball of iron 2 in. in diameter, weighing 1 lb , fastened to the end of the bar. Find the distance of the C. of G. of the pendulum from the top of the bar.

5. A cylinder of lead, 1 in. long, is cemented on to the end of a cylinder of wood 10 in. long, of the same cross-sectional area. Find the C. of G. of the body so formed, if the density of lead is 11·4 and that of wood 0·6.

6. A piece of wire, 10 in. long, is cut into two parts which are joined so as to form a letter T. If the upright part is 6 in. long, find the distance of the C. of G. from the bottom end. (Neglect thickness of the wire.)

7. A square board of side 1 ft. weighs 1 lb. If weights of 2 lbs., 3 lbs., 5 lbs. and 6 lbs. are placed in order at the angular points, find the C. of G. of the system.

8. Weights of 1 lb., 2 lbs., 3 lbs. are placed at the angular points of an equilateral triangle whose altitude is 1 ft. Find the distance of the C. of G. of the weights from the 1 lb.

9. A trapezium is made of a square ABCD of side 8 in., and a triangle BCE, whose base CE equals the side of the square. Calculate the distance of the C. of G. of the figure from the side DE.

10. Show, by using the theorem on p. 131, that the C. of G. of a triangle having its angular points at $x_1y_1$, $x_2y_2$, $x_3y_3$ is the point $\frac{x_1+x_2+x_3}{3}$, $\frac{y_1+y_2+y_3}{3}$.

**Centre of Gravity of a Trapezium.**

Let the parallel sides of the trapezium be of length $a$ and $b$, and let its altitude be $h$.

Draw the diagonal BD.

The weights of the triangles are proportional to their areas. Let $w_1$ be the weight of ADB, and $w_2$ the weight of BDC. Bisect AB at E, and DC at F.

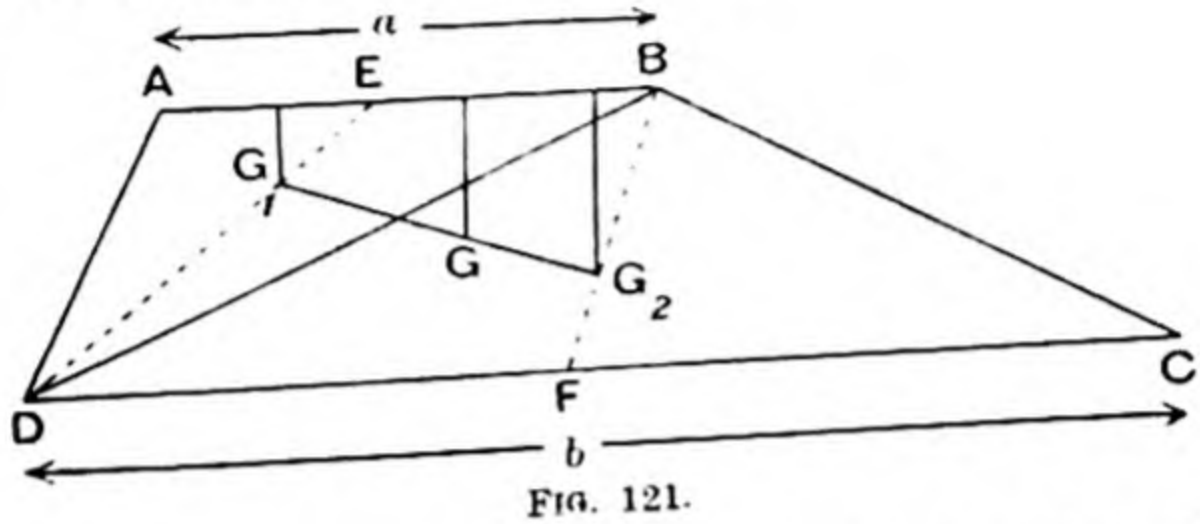



FIG. 121.

$w_1$ acts at $G_1$, the C. of G. of the triangle ADB, where

$$EG_1 = \tfrac{1}{3}ED.$$

$w_2$ acts at $G_2$, where $FG_2 = \tfrac{1}{3}FB$.

Suppose the plane of the trapezium to be horizontal, and take moments about the line AB.

Let $\bar{x}$ be the distance of G, the C. of G. of the trapezium from the line AB.

Then
$$w_1\frac{h}{3}+w_2\frac{2h}{3}=(w_1+w_2)\bar{x}.$$

But
$$\frac{w_1}{w_2}=\frac{\frac{1}{2}ah}{\frac{1}{2}bh}=\frac{a}{b};\quad \therefore\ a\frac{h}{3}+b\frac{2h}{3}=(a+b)\bar{x};$$

$$\therefore\ \bar{x}=\frac{h(a+2b)}{3(a+b)}.$$

If we divide the trapezium into strips parallel to AB, we see that the C. of G. of the trapezium G lies in the line EF.

Prove that
$$\frac{EG}{GF}=\frac{a+2b}{2a+b}.$$

The following examples illustrate problems in connection with the centre of gravity.

EXAMPLE 1. If a body is composed of two parts, and we know the position of the C. of G. of both parts, we can find the C. of G. of the complete body.

On AD, the side of a rectangle ABCD, an isosceles triangle of height 2 ft. is drawn. If AB = 3 ft., AD = 2 ft., find the C. of G. of the complete figure.

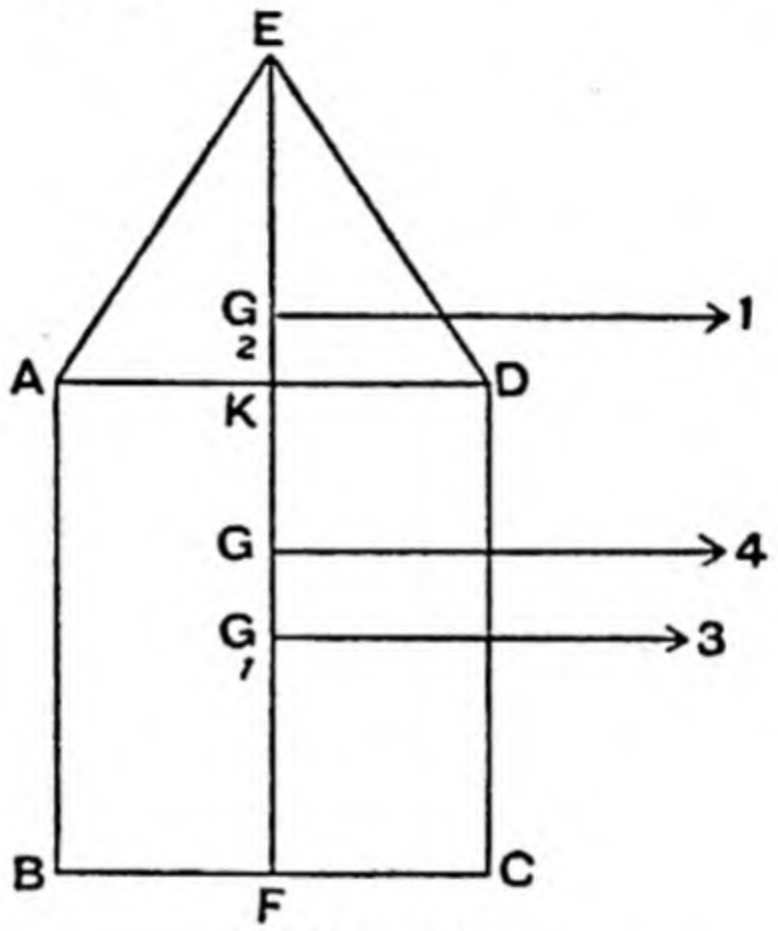



FIG. 122.

Imagine the plane of the figure to be horizontal, so that the weight of each part acts at right angles to the central line EKF.

The weights of the two parts of the figure are proportional to the areas, and are therefore in the ratio 3 : 1. They act at $G_1$ and $G_2$, where $G_1$ is the mid point of KF and $KG_2 = \frac{1}{3}KE$.

Since G is the point at which their resultant acts, we have, by taking moments about F,

$$1(G_2F) + 3(G_1F) = 4GF;$$

*i.e.* $$1(1\tfrac{1}{3}) + 3(\tfrac{3}{2}) = 4GF;$$

$$\therefore GF = 2\tfrac{1}{4} \text{ ft.}$$

EXAMPLE 2. If part of a body is removed, it is required to find the C. of G. of the remainder. when we know the C. of G. of the part removed and of the complete body.

From a circular disc, a circular part is removed, whose diameter equals the radius of the complete disc.

To find the C. of G. of the remainder.

The areas, and therefore the weights of the complete circle and of the part removed, will be in the ratio 4 : 1, so that if the weight of the part removed is taken as 1, the weight of the remainder will be 3. Suppose the part removed to be restored.

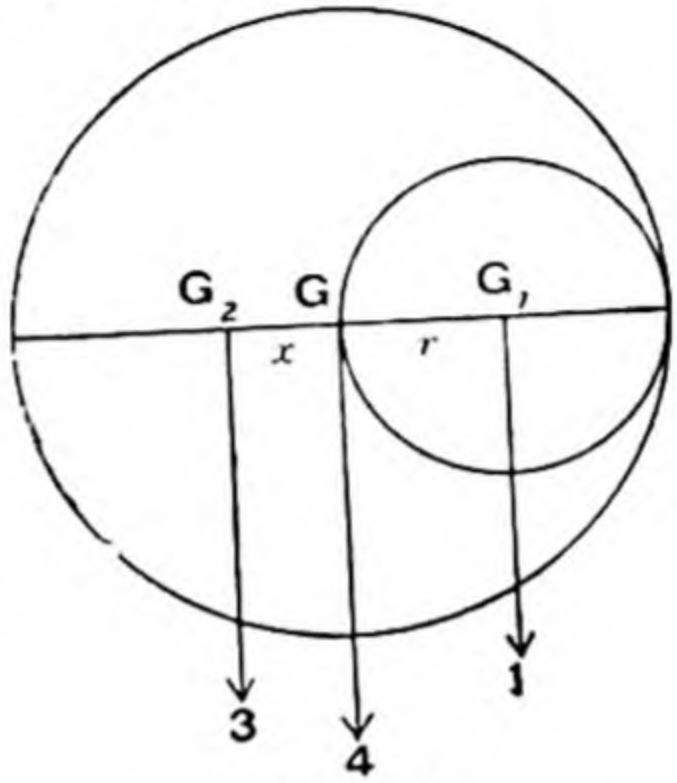



FIG. 123.

Then 4 is the resultant of 1 acting at $G_1$ and 3 acting at $G_2$ Therefore $G_2$ is on $G_1G$ produced, and if $G_2G$ be $x$ and $G_1G$ be $r$,

$$3x = 1 \cdot r; \quad \therefore \ x = \frac{r}{3}.$$

## EXAMPLES XXI.

**1.** Three uniform rods, AB = 5 in., BC = 12 in., CA = 13 in are jointed at their extremities to form a right-angled triangle ABC. Find the distance of the C. of G. of the three rods from BC. What would be the distance from BC of the C. of G. of the triangular lamina ABC?

**2.** A square has equilateral triangles described on three of its sides. Find the distance from the fourth side of the C. of G. of the figure so formed, the sides of the equilateral triangles being equal to the sides of the square.

**3.** A cylindrical can without a lid is 10 in. high, and the radius of the base, which is made of the same material, is 2 in. Find the distance of the C. of G. of the can from the base.

**4.** Two lines, parallel to the sides, are drawn through the centre of a square of side 10 in. One of the four squares so formed is removed. Find the C. of G. of the rest of the figure.

**5.** One of the six equal uniform rods which formed a regular hexagon is removed. Find the distance of the C. of G. of the remaining five from the original centre if the distance between two parallel sides was 2 ft.

**6.** Two diagonals of a square are drawn, and one of the triangles so formed is removed. Find the distance from the centre of the square of the C. of G. of the remaining figure, the side of the square being 6 in.

**7.** A square of $1\frac{1}{2}$ in. side is cut out of the middle of one of the longer sides of a rectangular plate whose sides are $4\frac{1}{2}$ in. and $2\frac{1}{2}$ in. Find the distance of the C. of G. of the remainder of the plate from the other long side of the rectangle. (Army.)

**8.** The top part of an isosceles triangle is removed by cutting it parallel to the base through the mid point of the perpendicular drawn from the vertex to the base. If the altitude was originally 8 in., find the distance, from the base, of the C. of G. of the remainder.

**9.** The end of a shed is a rectangle, 6 ft. wide, 12 ft. high, with an equilateral triangle above it. If the wind pressure on it is 50 lbs. to the square inch, find at what point the resultant pressure due to the wind will act.

**10.** The triangle formed by joining the middle points of the sides of a triangular lamina is removed; prove that the C. of G. of the remainder coincides with the C. of G. of the original triangle.

**11.** A circular disc of radius 1 ft. has two equal circular holes punched in it, the centres of these holes being at the mid points of two radii at right angles to one another. If the radius of each hole is 2 in., find the distance of the C. of G. of the remainder from the centre of the disc.

**12.** Two uniform rods OA, OB of the same material are rigidly joined at O, so that AOB is a right angle, and they are suspended by a string fastened at A. If AO = 12 in., BO = 8 in., find the inclination of AO to the vertical.

**13.** A right circular cone is cut by a plane parallel to the base which bisects the altitude. Find the distance of the C. of G. of the lower part from the base. C. of G. of a cone is one quarter the altitude from the base.

**14.** A solid is made of a cone which fits exactly on the top of a circular cylinder of the same material. If the height of the cone equals the height of the cylinder, find the distance of the C. of G. of the complete figure from the base.

**15.** A circular disc of iron, 2 ft. in diameter, is used as a flywheel. The C. of G. is found to be 0·1 in. from the centre. What should be the diameter of a hole drilled through the disc, 6 in. from the centre, to bring the C. of G. to the centre?

**16.** Plot out on squared paper the four points A(0, 0), B(0, 6), C(4, 6), D(5, 2). If the quadrilateral ABCD were lying on a horizontal plane, and C were slightly raised, find whether AB or AD would leave the plane. Give your reasons clearly.

**Stability.**

When a body is placed on a plane it will upset when the vertical through the C. of G. falls outside the contour of the base.

If the C. of G. of the body ABC is at G, this body, when placed on a horizontal plane, will upset, since the upward pressure of the plane on BC is unable to balance the weight of the body.

If, however, the body is placed on an inclined plane, on which it will not slip, the total reaction of the plane on BC can act at K, and will balance the force W if the vertical through G cuts the base at K.

It is clear that the limiting position of equilibrium is determined when the vertical through the c. of g. passes

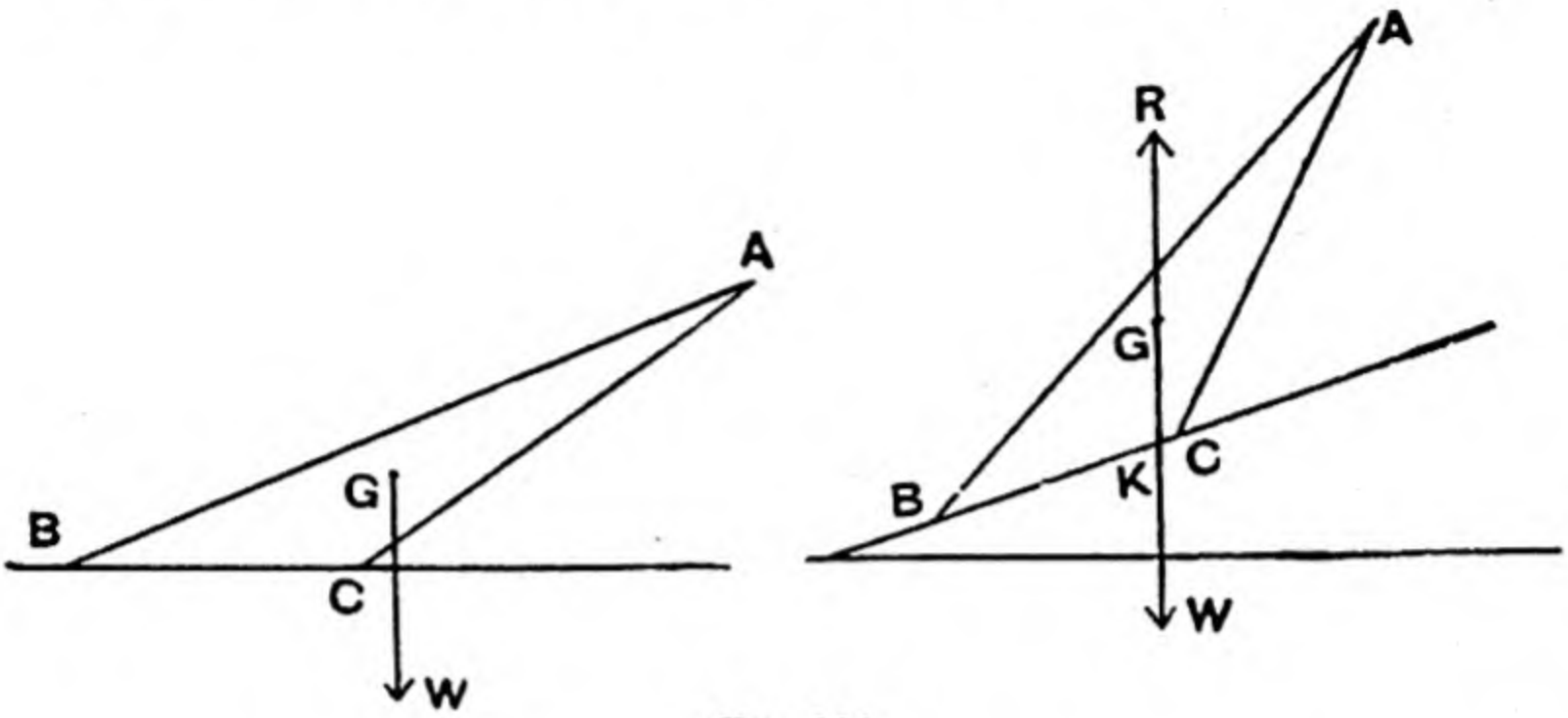



Fig. 124.

through the contour of the base, provided that the whole of the base is in contact with the surface on which the body is resting. An instructive example, when this condition is not fulfilled, is to consider how far a hollow circular cylinder may be pushed over the edge of a circular table before the cylinder will upset.

If a body is slightly displaced from its position of equilibrium and then left to itself, the forces acting on it may tend

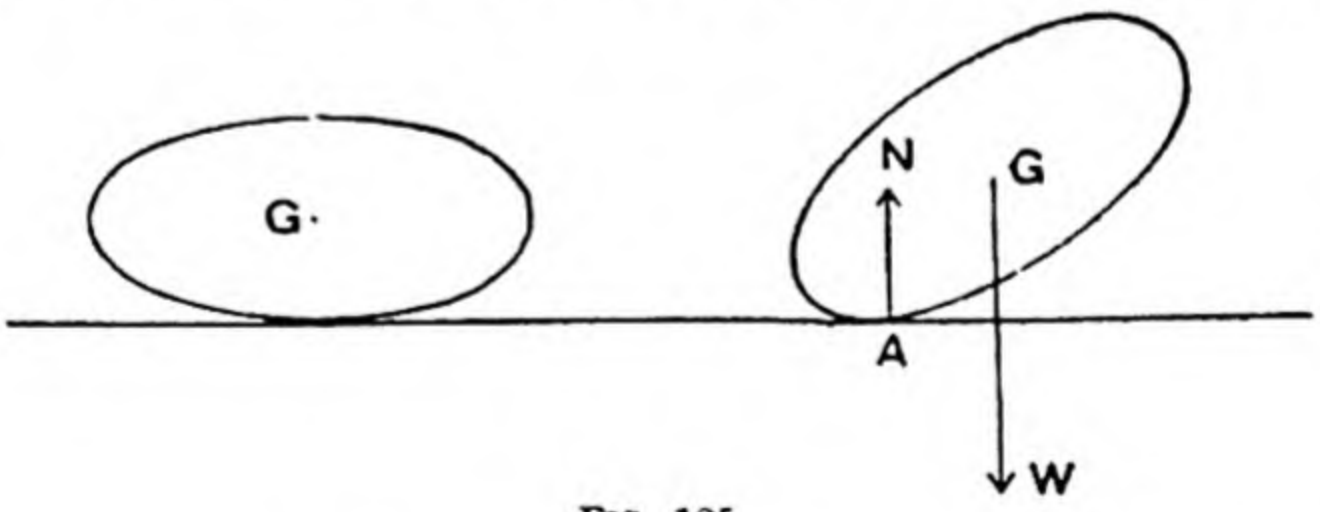



Fig. 125.

to make it return to its original position. In that case the body is said to be in *Stable Equilibrium.*

The moment of W about A causes a rotation of the body in Fig. 125, which will bring it back to its original position.

It will be noticed that, in the original position, the C. of G. of the body is as low as possible, and any displacement will raise the C. of G.

If, however, the forces acting on the body in its displaced position tend to take it further away from its equilibrium position, the body is said to be in *Unstable Equilibrium.*

In the original position the C. of G. of the body was as high as possible, and the displacement has caused the C. of G. to be lowered.

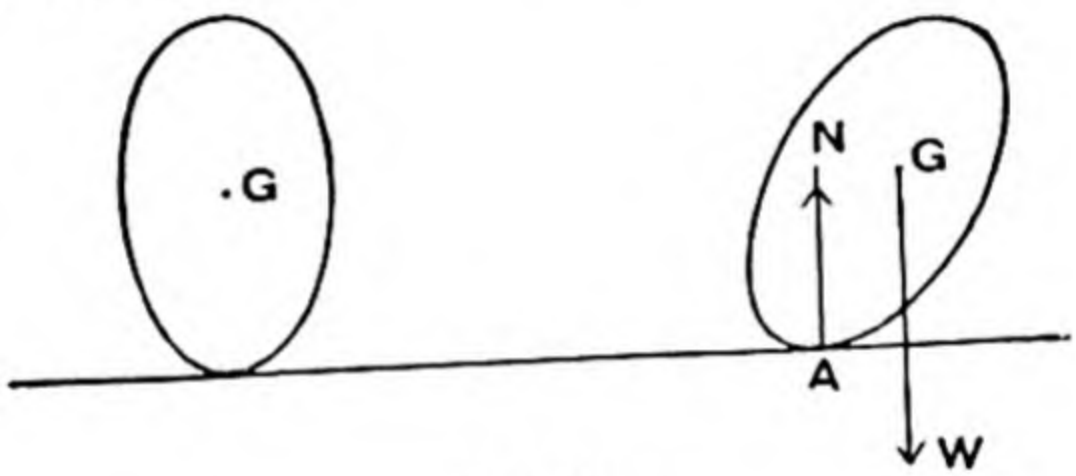



FIG. 126.

If the body, when displaced, remains in equilibrium in its new position, it is said to be in *Neutral Equilibrium.*

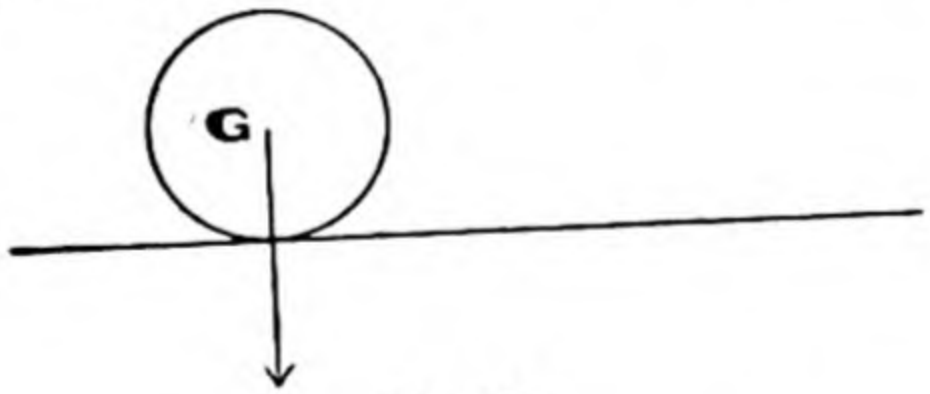



FIG. 127.

Supposing the cross-section of the body in the previous figures to be circular, a displacement produced by rolling the body on the plane will leave it still in equilibrium. In

this case the C. of G. of the body is neither raised nor lowered by the displacement.

It is possible for a body to be in stable equilibrium for one kind of displacement, but unstable for a different dis-

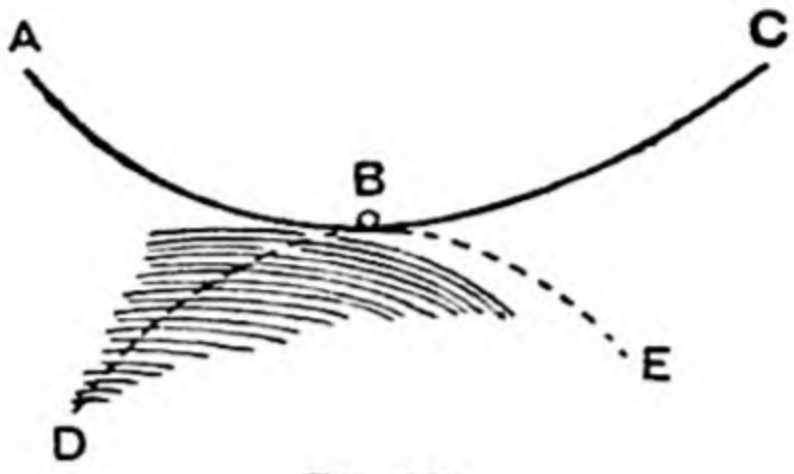



FIG. 128.

placement; for instance, a small ball at B on a saddle would be in stable equilibrium for a displacement in the plane ABC, but in unstable equilibrium for a displacement in the plane DBE.

EXAMPLE. A right circular cylinder is placed on a plane whose inclination to the horizontal is gradually increased.

To determine whether the cylinder will slip before it topples.

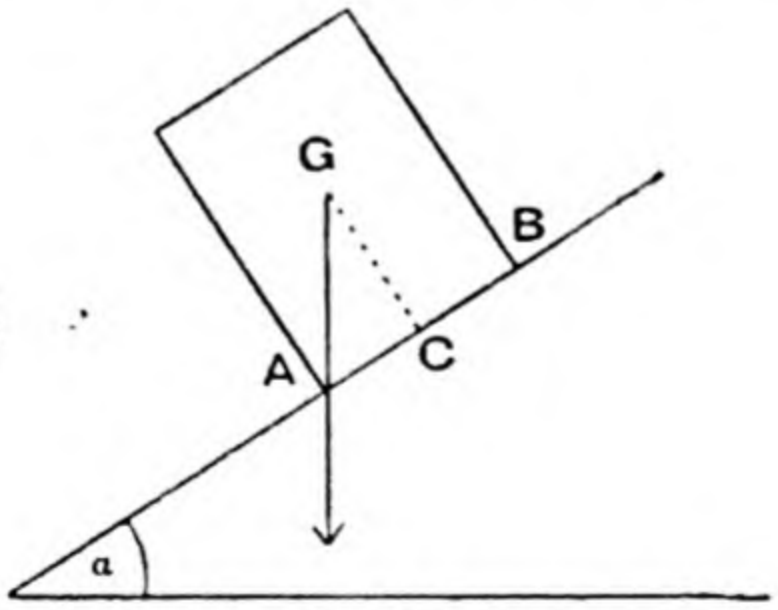



FIG. 129.

The cylinder will topple if the friction is sufficiently great to keep A from moving until the angle of the plane is increased sufficiently for G to come vertically above A.

If $r$ is the radius of the base and $h$ the altitude of the cylinder,

$$\tan \hat{\mathrm{AGC}} = \frac{r}{\frac{h}{2}} = \frac{2r}{h},$$

and, by geometry, $\mathrm{AGC} = \alpha$; $\therefore \tan \alpha = \frac{2r}{h}$.

The cylinder will slide on the inclined plane when the tangent of the angle of the plane becomes $= \mu$, the coefficient of friction.

If, therefore, $\mu < \frac{2r}{h}$, the cylinder will slide before the inclination of the plane is made as great as $\alpha$. Therefore the cylinder will slide before it topples.

If $\mu > \frac{2r}{h}$, the inclination of the plane would have to be greater than $\alpha$ to make the cylinder slide. Therefore the cylinder will topple before it slides.

## EXAMPLES XXII.

**1.** A cube is placed on a rough inclined plane with two edges of its base parallel to a line of greatest slope. Find the angle of the plane if the cube is just about to topple.

**2.** A hollow cylinder is placed with its base on a rough plane inclined to the horizontal at 40°. If the radius of the base is 4 in., find the greatest height the cylinder can have without upsetting.

**3.** A triangular lamina is placed in a vertical plane, with one of its sides 2 in. long, on a horizontal plane. If it is just on the point of toppling on to another side, 3 in. long, find the length of the third side.

**4.** A brick is placed upon an equal brick which projects over the edge of a table, the ends of the bricks being placed parallel to the edge of the table. Show how the bricks must be placed so that the upper one is projecting as far as possible beyond the edge of the table. Extend your result to the case of three bricks.

**5.** How many bricks would be required in Qu. 4 before the top brick projected entirely beyond the vertical through the edge of the table ?

**6.** A hollow cylinder whose internal radius is 1 ft. is placed on a horizontal circular table, whose radius is 4 ft. How far can the cylinder project over the edge of the table ?

**7.** A square table stands on four legs placed at the mid points of the sides. If its weight is W, find the weight of a man who just upsets the table by sitting at one corner.

**8.** A table consists of a uniform circular board weighing 9 lbs., supported by three vertical legs fixed at equal distances round the circumference. A weight of 12 lbs. placed on the edge of the table, midway between two legs, is just sufficient to cause the table to overturn. Find the weight of each leg.

**9.** BAC is a triangular board, $\hat{BAC}$ being a right angle. The board rests in a vertical plane with AC on a horizontal table. D is the mid point of AC. Show that, if the triangle BAD is removed, the remainder will be on the point of toppling over.

**10.** A uniform square board weighing 40 lbs. is suspended from a point by means of four equal threads (of length equal to the diagonal of the square), attached (*a*) to the four corners, (*b*) to the mid points of the sides. Find the tension in the threads in each case. What is the effect of cutting one of the threads ?

## Laboratory Experiments.

1. Find the C. of G. of a triangular piece of cardboard, and show that it divides the medians in the ratio 2 : 1.

2. Find the C. of G. of a wire as on p. 138.

3. Take a rectangular piece of stiff cardboard. Through holes in the cardboard pass strings carrying weights, and knot each string on the other side of the cardboard, so that the weights are in contact with the cardboard. Hold the board in a vertical plane with one edge horizontal. Take this as the X axis and calculate $\bar{x}$.

Turn the board through a right angle in a vertical plane and calculate $\bar{y}$.

Show that the board balances at the point given by $\bar{x}$, $\bar{y}$ when supported at this point, so that the board is horizontal.

## MISCELLANEOUS EXAMPLES.

### CHAPTERS I.–V.

### H.

**1.** A uniform rectangular board ABCD, weighing 3 lbs., can turn in a vertical plane round a pin at A. It is kept in equilibrium, with AB horizontal, by a smooth peg at E, to the left of C, on which the lower edge CD rests. If AB = 2 ft., BC = 1 ft., CE = 2 in., find the magnitude and direction of the action of the pin at A.

**2.** Weights of W and 2W are placed in two adjacent angles of a square. Find the weights that must be placed in the other angles, so that the C. of G. of the four weights may be at the centre of the square.

**3.** Two books, similar in every respect, each 10 in. long, lie one exactly upon the other on a table, over the edge of which they project 3 in. How much further may the upper book be pushed out before they fall over?

**4.** A uniform rod, 4 ft. long, weighing 2 lbs., has one end on the ground and rests in a vertical plane against a smooth horizontal rail, which is 3 ft. from the ground. If the rod makes an angle of 30° with the vertical, find the resultant reaction at the ground and the frictional force which is preventing the rod from slipping.

**5.** A force acting up a rough plane inclined at 40° to the horizon just supports a weight upon it ($\mu = 0{\cdot}2$). Find the inclination of a smooth plane, such that the same force acting up it shall support the same weight.

### I.

**1.** A string fastened to a nail, and hanging vertically, supports the apex of a uniform isosceles triangle of weight W, which rests with its base upon a smooth horizontal plane; find the tension of the string and the pressure on the plane. (The triangle is not vertical.)

2. A number of penny pieces are cemented together, so that each just projects over the one below it by one-fifth of its diameter. Find how many may thus be piled without falling, when the lowest is on a horizontal plane.

3. The following table gives the relation between the load and effort for a machine:

| | | | | | |
|---|---|---|---|---|---|
| Load, - | 56 | 112 | 168 | 224 | 340 lbs. |
| Effort, - | 10 | 17 | 24 | 28 | 42 lbs. |

Find the law of the machine, and find its efficiency in each case if the velocity ratio is 9.

4. The figure represents the section of a girder. Find the position of its C. of G.

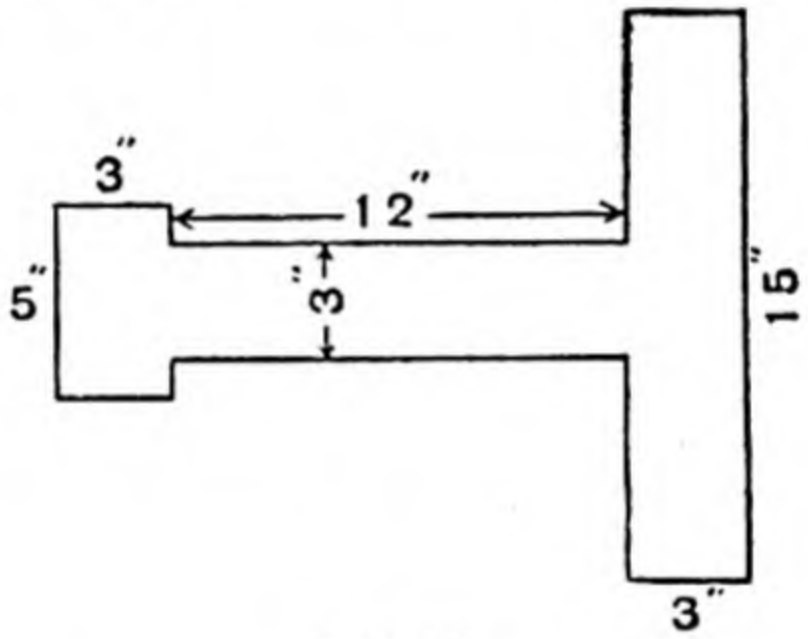



FIG. 131.

5. A weight of 4 lbs. rests on a rough plane, whose inclination to the horizon is 30°. A string attached to the weight passes along the plane and over its lower edge, and has a weight of 4 ozs. fastened to it. Find the ratio of friction to the pressure on the plane.

J.

1. An equilateral triangular lamina, of side 1 ft., rests in a vertical plane, with one side horizontal and the other two sides supported by two smooth pegs 8 in. apart on the same level. If the lamina weighs 1 lb., find the pressure on the pegs.

**2.** A cube of weight 10 lbs. resting on a horizontal table is gradually raised round one edge by a smooth wedge pressed under it. When the cube has been turned through an angle of 20°, which is half the angle of the wedge, find the friction between it and the table, at the edge on which it turns.

**3.** A cube is placed on a plane slightly inclined to the horizontal, so that two faces of the cube are in vertical planes. If the coefficient of friction is $< 1$, prove that, as the plane is gradually tilted, the cube will slide before it topples over.

**4.** A uniform wire is bent into the shape of an isosceles triangle, whose equal sides are 1 ft. and base angles 70°. Find the distance of the C. of G. of the wire from the base.

**5.** If a solid cone, whose height is twice the diameter of its base, be suspended from a point in the circumference of its base, prove that in the position of equilibrium its axis will be inclined at 45° to the horizon. (See p. 145, Ex. 13.)

## K.

**1.** One end B of a uniform rod AB is jointed to a point B in another uniform rod CD. A and D are joined by a string, so that ABD is a right angle. The system hangs from A, round which it can turn. If $AB = BC = 4$, $BD = 8$, and if the weight of AB is double the weight of CD, find the position of equilibrium.

**2.** Find the altitude of the largest cone, upon a base of diameter $d$, that will just stand upon a rough plane inclined at $\alpha$ to the horizon. Find also the highest point on the side to which any weight may be fastened without upsetting the cone.

**3.** A uniform lamina in the form of a right-angled isosceles triangle is placed with its longest side in contact with and along a line of greatest slope of a rough plane inclined to the horizon at 30°. If $\mu = 0{\cdot}6$, show that the lamina will remain in equilibrium.

**4.** If the lamina in Qu. 3 is pulled up the plane by a string attached to its highest point, and parallel to the line of greatest slope, will the lamina tilt or slide when the tension of the string is increased sufficiently to move the lamina ?

5. The bar AB is free to turn about B in a horizontal plane. The horizontal bar CD, perforated at equal distances, is attached

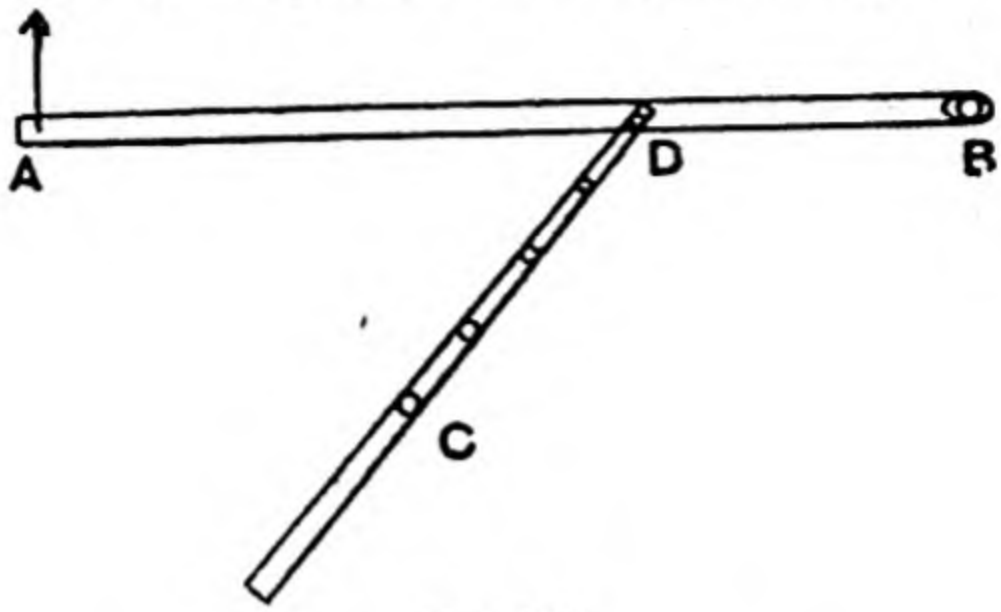



FIG. 132.

to AB at D by a universal joint, and is held in position by a fixed pin C passing through one of the holes. If a horizontal force of 10 lbs. wt. is applied at A perpendicular to AB, find the stresses produced at B, C and D, given AD = 15 in., DB = 5 in., DC = 5 in., $D\hat{B}C = 30°$. (Neglect friction.)

L.

1. A rectangular lamina, 2 ft. by 1 ft., weighing 2 lbs., is suspended at the mid point of one of its long edges, and a weight of 4 ozs. is hung at one end of this edge. Find the inclination of the lamina to the horizontal in the position of equilibrium.

2. A light rod AB, 4 ft. long, can turn in a vertical plane about a smooth hinge at A, and is kept in a horizontal position by a light rod CB, C being fixed $1\frac{1}{2}$ ft. vertically below A. If a weight of 10 lbs. is fastened to AB at D, where AD = 1 ft., explain why the thrust of CB at B acts along CB, and find its magnitude.

3. ABCD is a square plate. It is in equilibrium under the action of four forces, two acting at A, one at B, and one at C. The forces at A are 10 lbs. and 5 lbs. along the direction DA and BA respectively, the force at C is along the line DC. Find the magnitude and direction of the force at B.

4. A weight of 20 lbs. is placed on a rough plane inclined at 20° to the horizon. It is found that the least force acting

downwards, along the slope of the plane, which will cause it to move is 2·4 lbs. Find the least force, acting upwards along the slope, required to pull it up, and find at what angle the plane must be inclined so that the weight will just run down of its own accord.

5. A cone of given vertical angle $2\alpha$ rests on a rough plane which is inclined to the horizon. As the inclination of the plane is increased, show that the cone will slide before it topples over if $\mu$ be less than $4 \tan \alpha$.

## M.

1. A rectangular piece of cardboard ABCD can turn freely round O, the mid point of its side AB. AB = 12 in., BC = 8 in. When a weight of 10 grams is fastened to B, the side AB makes 20° with the horizontal. Find the weight of the cardboard.

2. A uniform rod weighing 150 grams rests with one end on a smooth horizontal floor, while the other end is supported by a string, so that the rod makes an angle of 20° with the floor. Find the direction of the string and its tension.

3. A large rectangular box contains a uniform spherical ball weighing 5 lbs. If the box is placed upon a horizontal table and then tilted round one of its edges through 25°, find the pressure of the ball on the two faces of the box which support it.

4. The diagram shows the vertical section of a body which contains the C. of G. The quadrants represent a section of iron which is nine times the density of the rest. Find the distance of the C. of G. from the left edge.

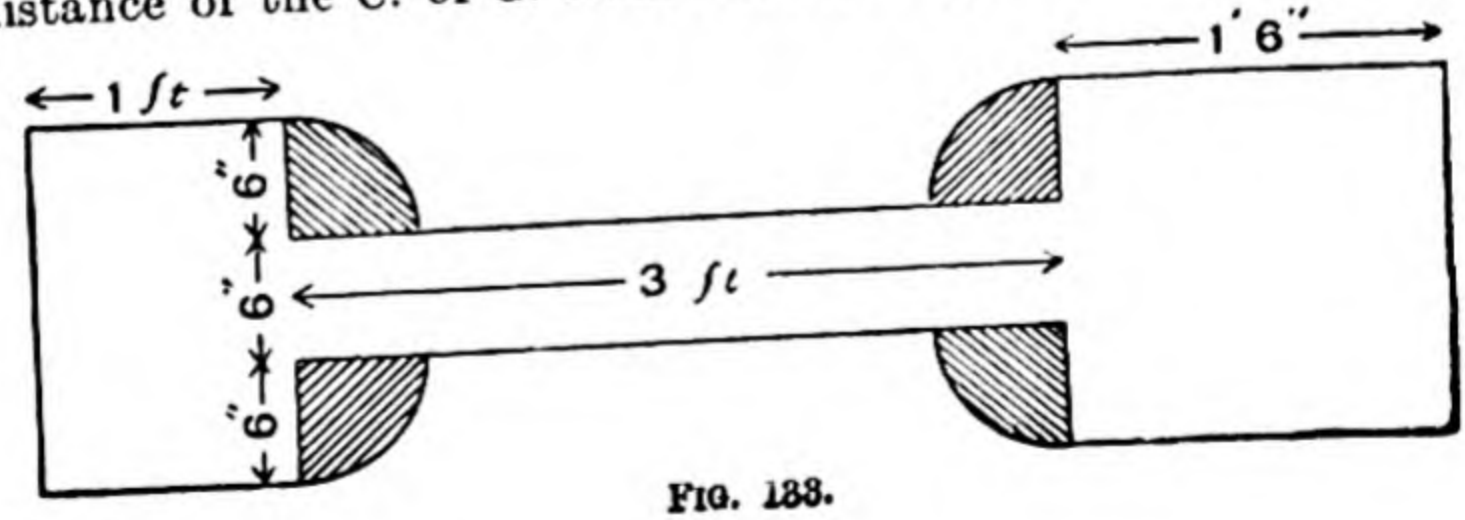



FIG. 133.

5. A uniform smooth rod rests on two smooth inclined planes at right angles to their common line of intersection. If the angles of the planes are 30° and 45° respectively, find the inclination of the rod to the horizontal.

## N.

1. A gate is supported by two hinges in such a way that the weight of the gate is carried entirely by the lower hinge, so that the action of the upper hinge is entirely horizontal. The distance between the hinges is 3 ft. and the weight of the gate, 60 lbs., acts along a vertical line $3\frac{1}{2}$ ft. from the line of the hinges. Find the force exerted by each hinge.

2. A uniform circular disc of weight W is supported in a vertical plane on two smooth pegs, so placed that the line joining them subtends a right angle at the centre of the disc, and makes an angle of 30° with the horizontal. Find the pressure on each peg.

3. A heavy beam is supported by a knife edge at A and a spring balance at B. The beam carries a load at some point P between A and B. and the reading of the balance is then $R_1$ lbs. When the load is moved along the beam a distance $x$ ft. to the right, the reading of the balance is $R_2$. If $l$ ft. be the length of AB, find an expression for the weight of the load.

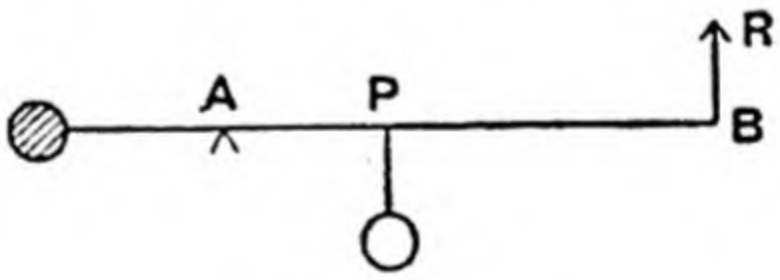



FIG. 134.

The load being in its initial position, the knife edge is moved a distance $y$ ft. to the right and the balance reading is now $R_3$. Find an expression for the combined weight of the load and the beam, and prove that the weight of the beam is

$$R_3+\frac{l}{y}(R_1-R_3)+\frac{l}{x}(R_1-R_2). \qquad \text{(Army.)}$$

**4.** A and B are the centres of the two wheels of a bicycle considered symmetrical about a plane through both wheels. G is the C. of G. of the bicycle. When both wheels are on the

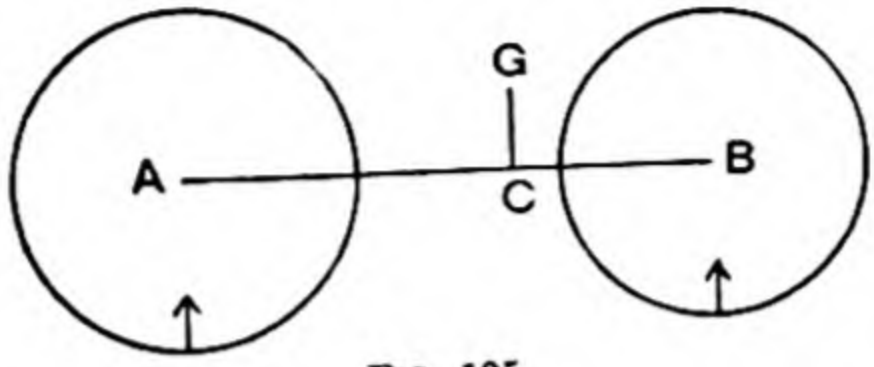



FIG. 135.

same level the pressures of the wheels A and B downwards are $W_A$, $W_B$ respectively. When A is jacked up a vertical height $h$ above B, the vertical pressures are now $W_\alpha$, $W_\beta$.

Prove that $$GC = \frac{d}{W}(W_A - W_\alpha)\frac{\sqrt{d^2 - h^2}}{h},$$

and that $CB = \frac{W_A}{W}d$, where $W$ = total weight of the bicycle and $AB = d$.

**5.** A solid sphere has a fine bore made through it along a chord subtending an angle $2\alpha$ at the centre, and is free to move on a vertical rod fitting loosely into the bore. Show that the sphere may rest in equilibrium with no further support if the angle of friction is greater than $\alpha$.

# REVISION QUESTIONS

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**1.** A uniform bar AB, 10 ft. long, weighing 40 lbs., rests on supports at C and D, where AC = 2 ft., AD = 9 ft. What weight must be placed at B to make the pressures at C and D equal to one another ?

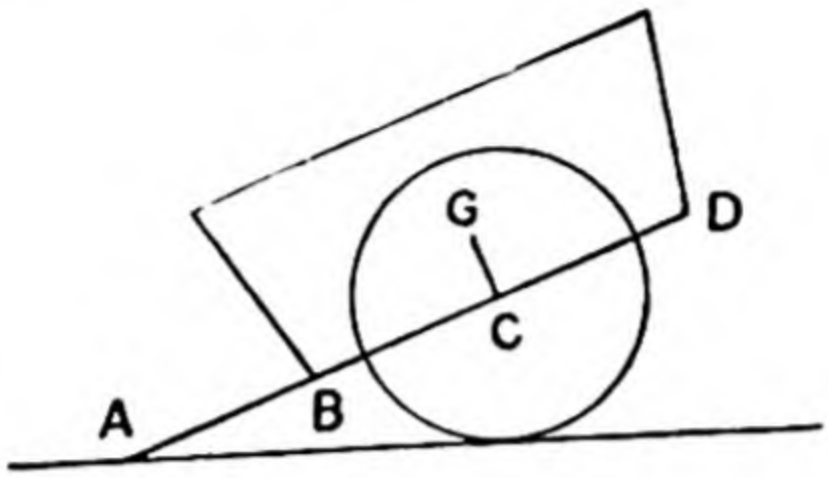
**2.** A toboggan weighing 60 lbs. is placed so as to run down a hard snow slope. The length AB in contact with the ground is $2\frac{1}{2}$ ft., and when on the level the C. of G. is 6″ vertically above the mid point of AB. When on a slope of 1 in 10 the toboggan just runs down. Determine the position of the resultant force due to the surface, and find the frictional force opposing motion.

**3.** A workman on a scaffold raises a load of 1 cwt. by pulling upwards on a rope which passes round a pulley in a block to which the load is attached. The rope goes round another pulley in a block fastened to the scaffold, then round a second pulley in the lower block, and is fastened to the bottom of the upper block. Find the velocity ratio of this system and its efficiency if an effort of 42 lbs. weight is required to raise the load.

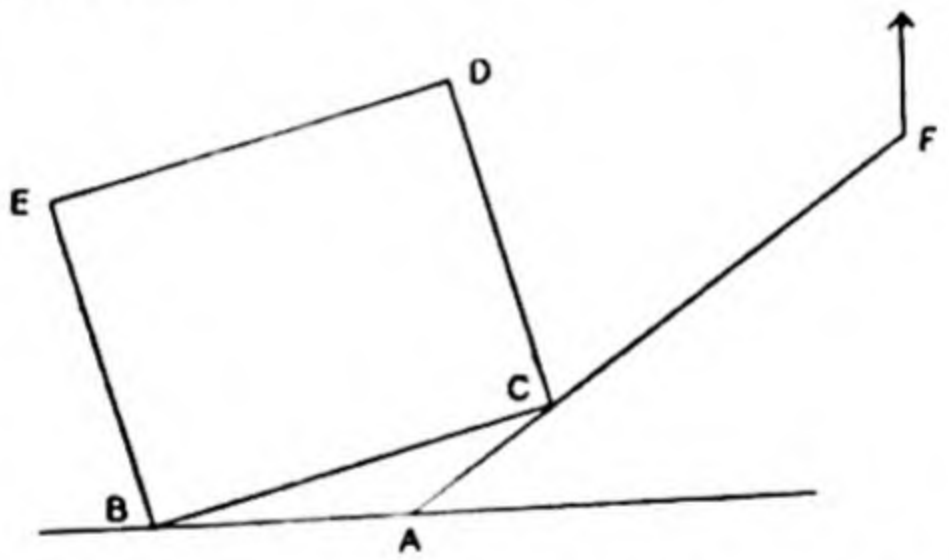
What effort would be required if the system were inverted, and the load, after being fastened to the block containing one pulley, were lifted by a man standing on the ground ? (Assume the efficiency to be unaltered.)

**4.** The diagram represents a child's cart with two wheels and two handles, the total weight being 20 lbs. The C. of G. is at G, 4″ from the mid point of the axle, GC being perpen-

dicular to AD. The radius of the wheel is 9″ and C the centre may be taken on the line ABD. If AB = BC = 1 foot, find what vertical force at the handles A and A′ will be required to lift them.




**5.** A cubical block of stone whose edge is 2 ft. is being tilted by a uniform crowbar 5 ft. long whose weight is 10 lbs.



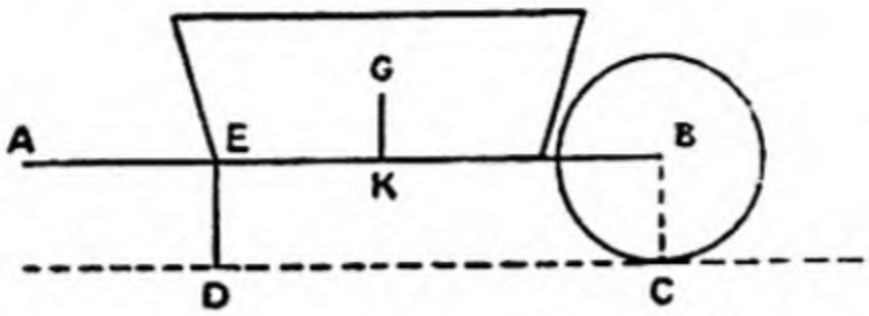

When the edge C of the stone is 6″ above the ground the distance AC = 10″ and the vertical force at F is 30 lbs. wt. Find the pressure of the stone on the crowbar at C assuming its direction to be perpendicular to AF.

**6.** A rectangular block of stone weighing $1\frac{1}{2}$ tons is 5 feet high and stands on its end which is 2 feet square. It is then tilted about one of the edges of the base until it has turned through 20°. If let go, find whether it will turn over or fall back again.

Find the horizontal force acting at right angles to the highest edge at its mid point which will keep the block in equilibrium in the tilted position.

**7.** The diagram represents a wheelbarrow of weight 1 cwt. AB is horizontal and the legs are vertical. The C. of G. is at G, 6″ above AB where EK = $1\frac{1}{2}$ ft., KB = $2\frac{1}{2}$ ft., ED = BC = 1 ft.

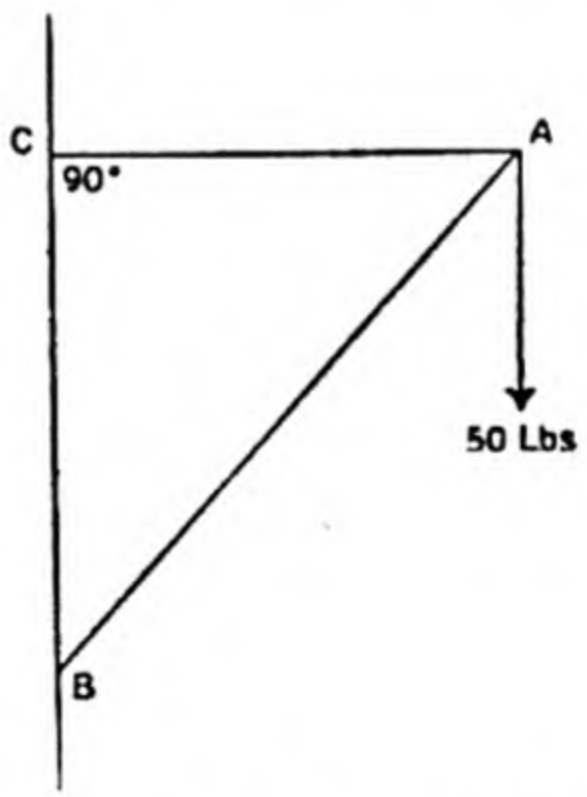
Find the pressure on the ground at C.




If the friction at the axle can be neglected, find graphically, or otherwise, the normal pressure at C and the direction of the forces exerted by the legs on the ground when the barrow is on a slope which brings C 1 ft. above D. (Consider all forces to be in the plane of the paper and then divide the force at D between the two legs.)

If the barrow is then on the point of sliding backwards state (with reasons) whether or no it would slide if turned round so that C is below D.

**8.** A load of 50 lbs. is supported by a bracket of light rods, AC, AB, hinged to a wall at C and B. If AB = 6 ft., AC = 4 ft., find the thrust in AB.



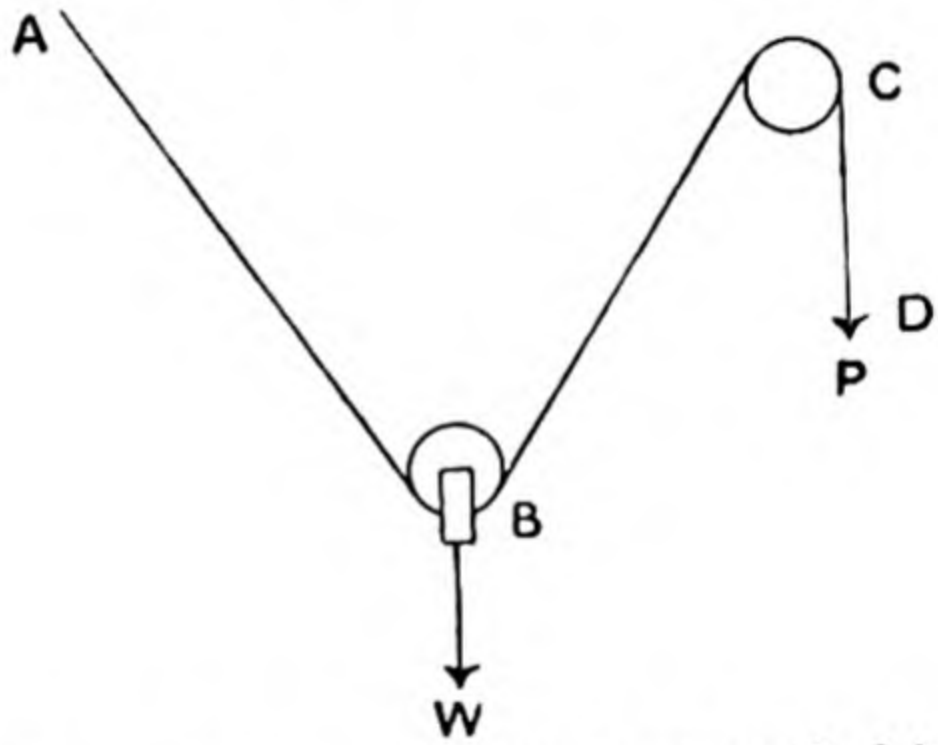

If a rope is fastened to A and to a point D in the wall 2 ft. vertically above C and then strained until its tension is 30 lbs.,

find whether the rod AC is then in a state of tension or compression, and explain carefully on what grounds you make your decision.

**9.** Four bars OA, OB, OC, OD, forming part of a girder, are jointed together by a pin at O. AO thrusts vertically upwards at O, OB pulls horizontally to the right, CO and DO thrust downwards towards O, the angles BOC, COD both being 60° and OC between OB and OD. If the force exerted by each bar is 1 ton, find what other force must act at O to preserve equilibrium.

**10.** A sphere of radius 1 ft. and weight 5 lbs. is placed on two smooth parallel horizontal rails. Find how far apart the rails should be placed so that the pressure on each should equal the weight of the sphere.

**11.** A cord fastened to a point A, supports a pulley at B and passes over a fixed pulley at C. A weight W of 2 lbs.




fastened to B is kept in equilibrium by a vertical force P at D. If friction is neglected prove that the part of the cord, BA and BC, will make equal angles with the vertical.

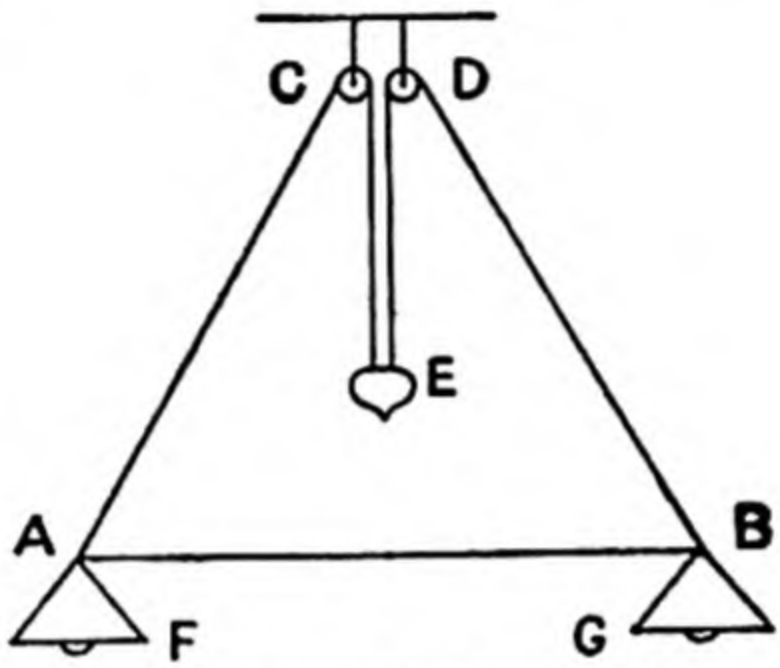
If the angles are 30° find P, taking the pulley B to weigh $\frac{1}{2}$ lb.

**12.** Two small frictionless pulleys A and B are fixed in a horizontal line 4 ft. apart. A string CABD passes over A, through the smooth ring of a 5 lb. weight, and then over B. The ring is kept in equilibrium 2 ft. below AB by vertical

forces applied at the ends, C and D, of the string. Find these forces and determine also their magnitudes if the ring were 1 ft. higher. If the weight is slowly raised between these two positions, find the work done by the forces at C and D.

How would the value of these forces be affected if the portions of string CA and BD were inclined outwards at 45° to the vertical, but kept in the same plane.

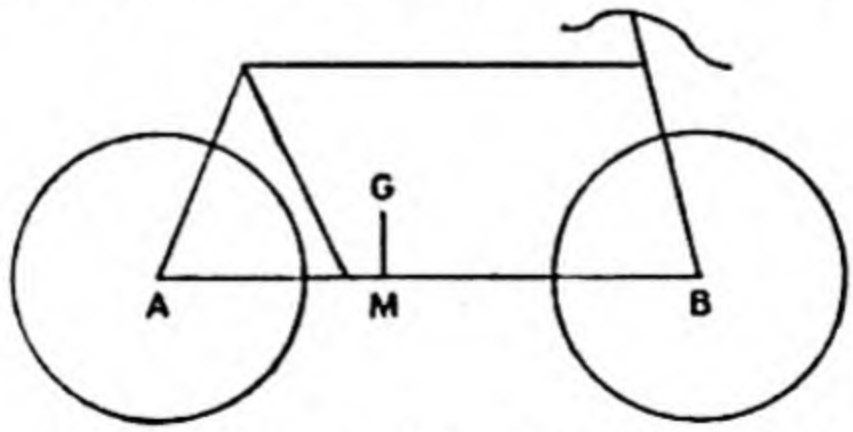
**13.** Two electric lamps are suspended by cords passing from A and B over small pulleys at C and D and fastened to a counterpoise E.




A and B are kept apart by a light rod whose weight may be neglected. If F and G weigh $\frac{1}{2}$ lb. each, find the weight of E necessary to maintain equilibrium if friction is negligible when $AB = 3\frac{1}{4}$ ft.; $AC = BD = 3$ ft.; $CAB = DBA = 60°$.

Find also the thrust in the rod AB.

**14.** The diagram represents a bicycle of weight 50 lbs. whose c. of G. is at G. GM is perpendicular to AB and $GM = 6''$,




$AM = 20''$, $MB = 28''$. If a cord is fastened to the rim of the front wheel and pulled vertically upwards until the wheel is

lifted 1 ft., find the pressure then exerted by the back wheel on the ground.

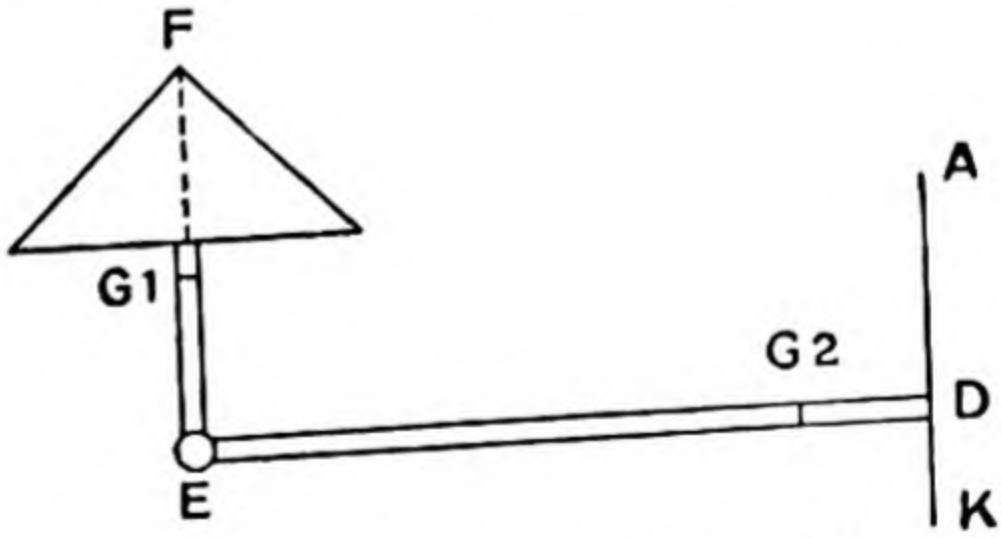
**15.** A string AB, 20 cms. long, is fastened to a point A on the surface of a sphere whose radius is 30 cms. The sphere is then placed on a smooth plane inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$, and the end B of the string is fastened to a point on the surface of the plane. Prove that in the position of equilibrium the string will be horizontal.

Assuming this to be proved, find the tension in the string if the sphere weighs 2 lbs.

**16.** A uniform thin hemispherical glass bowl of radius $\frac{1}{2}$ ft. and weight 20 ozs. rests with its convex surface on a smooth table. A vertical force of 9 ozs. applied at a point on the rim keeps the bowl tilted through 42°. Find the position this experiment gives for the centre of gravity of the bowl.

**17.** A uniform circular table has three similar legs at A, B, C, 3 ft. from one another and $2\frac{1}{2}$ ft. long. When any leg (say at A) is lifted from the ground the table can just be balanced when it has been tilted through 25°. Draw a vertical section through the median from A of the triangle ABC in the tilted position, and hence find the position of the C. of G. of the table.

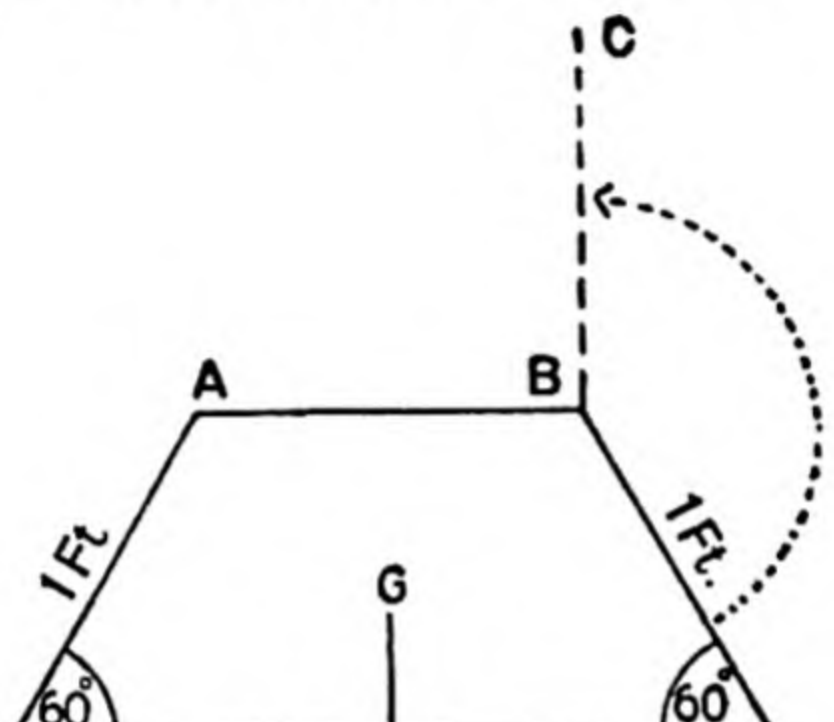
**18.** The diagram represents a reading lamp fastened to a hook in a smooth wall at A. The stand is an equilateral triangle, of which D is the mid point and A the vertex. ADK




and EF are vertical. The resultant thrust of the wall acts at K, the mid point of the base of the equilateral triangle. The stand with ED weighs 2 lbs., its C. of G. being $G_2$, the part

EF weighs $\frac{1}{2}$ lb., and its C. of G. is $G_1$; E is 1 ft. from the wall. $AK = 6''$, $EG_2 = 10''$, $EG_1 = 3''$. Find the C. of G. of the whole lamp in this position. By drawing or otherwise find the direction and magnitude of the force at A.

**19.** The diagram is a side view of a coal-box with a uniform square lid BC whose weight is one-sixth of the weight of the whole box. When the lid is shut the C. of G. of the box is at G, 4″ above the centre of the base. Find the position of the C. of G. of the rest of the box, excluding the lid.




If the lid is opened until it is vertical (BC′) show that the C. of G. of the whole box moves to a point whose distance from G is one-sixth of the distance between the two positions of the centre of the lid.

**20.** A two-foot rule is hinged at its middle and placed upon a horizontal table. One half is kept fixed while the other is rotated round the hinge. Prove that the C. of G. of the complete ruler describes a semicircle.

# ANSWERS TO QUESTIONS IN THE TEXT.

CHAP. I. page 4, line 16. 300 grams wt. To right.
17. 300 grams. wt. To left.
19. T = 300 grams wt. to the right
and T = 300 grams wt. to the left.
5, last line. T = W.

CHAP. III. page 26, line 10. No.
12. A straight line through B at right angles to OB.
14. Yes.

CHAP. VI. page 45, line 3. Friction being the same, the weight would be less, as its distance from the axle is greater.
46 3. $T_1 + T_2$ + wt. of pulley.
5. Increased, as there is greater pressure on the axle.
13. $2x$.
48 10. The work done to raise the block becomes a smaller fraction of the total work done on the machine.

CHAP. VIII. page 57, line 18. The load would not rise.
122 16. It must be in the same straight line as W.

## CHAPTER VI.

### PARALLEL FORCES. GRAPHICAL METHOD.

It has been shown in Chap. I. p. 33, that the line of action of the resultant of two like parallel forces acting at C and D cuts the line CD at a point O, such that $P \cdot CO = Q \cdot DO$. We shall now give a graphical method of finding the position of this resultant based upon the Triangle of Forces.

**To find the Line of Action of the Resultant of two like Parallel Forces acting on a Rigid Body.**

Let P and Q be two unequal parallel forces acting on a rigid body in the same direction (they are then said to be "like" forces).

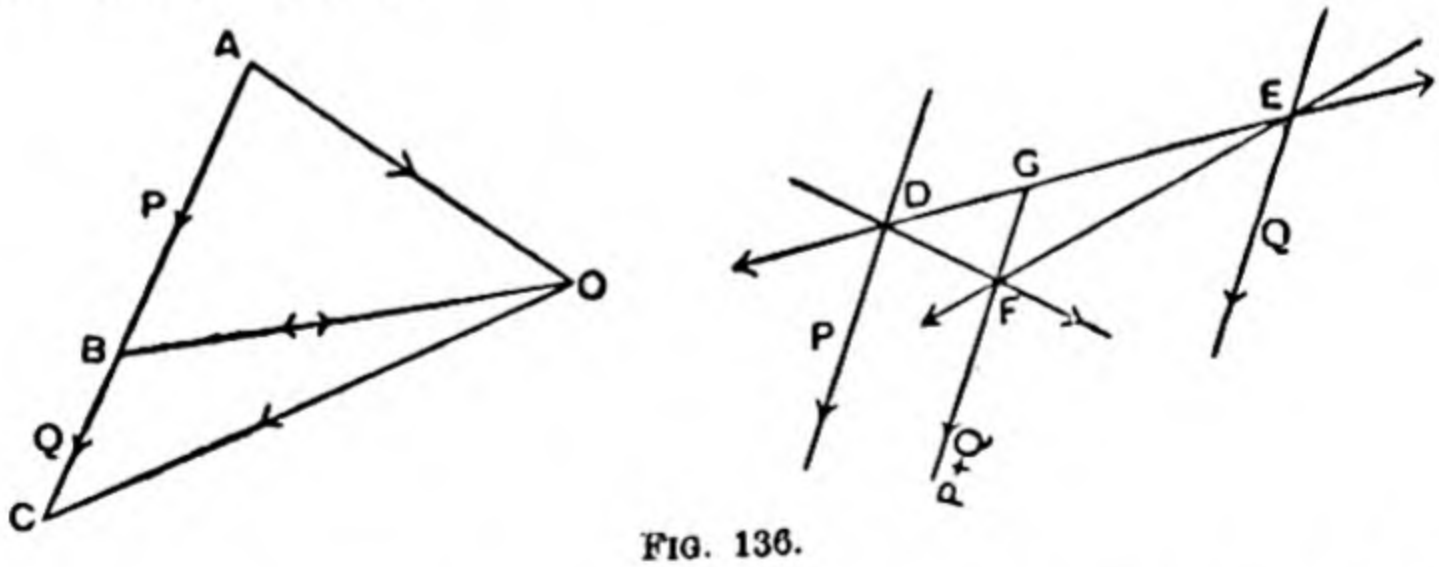



FIG. 136.

Draw a line ABC parallel to P and Q, and mark off AB to represent P and BC to represent Q.

Take any point O. Join AO, BO, CO.

Draw a line DE parallel to BO to intersect the lines of action of P and Q at D and E respectively.

*Note that* BO *is the line common to both triangles* ABO *and* BOC.

By the triangle of forces ABO, P acting at D may be replaced by forces represented by AO and OB.

Similarly, Q acting at E may be replaced by forces represented by BO and OC.

Let the forces represented by AO and OC intersect at F.

The forces represented by OB and BO will act along the line DE in opposite directions, and will neutralise one another.

The forces represented by AO and OC, acting at F, will have as their resultant a force represented by AC (from the triangle of forces AOC), *i.e.* a force of magnitude P + Q.

Therefore the original parallel forces P and Q will be equivalent to a force P + Q parallel to P and Q. This force will cut DE at G.

To find the position of G.

The triangles DGF and ABO are similar ;

$$\therefore \frac{DG}{GF} = \frac{BO}{AB}, \quad i.e.\ AB \,.\, DG = BO \,.\, GF.$$

Also the triangles EGF and OBC are similar ;

$$\therefore \frac{EG}{GF} = \frac{OB}{BC}; \quad \therefore\ BC \,.\, EG = OB \,.\, GF;$$

$$\therefore\ AB \,.\, DG = BC \,.\, EG, \quad i.e.\ P \,.\, DG = Q \,.\, EG.$$

The figure AOC is called a Link (or Funicular) Polygon.

**Unlike Parallel Forces.**

If the forces P and Q are unlike, we draw AB to represent P, and BC in the opposite direction to represent Q.

As before, we draw DE parallel to BO to cut the lines of action of P and Q at D and E.

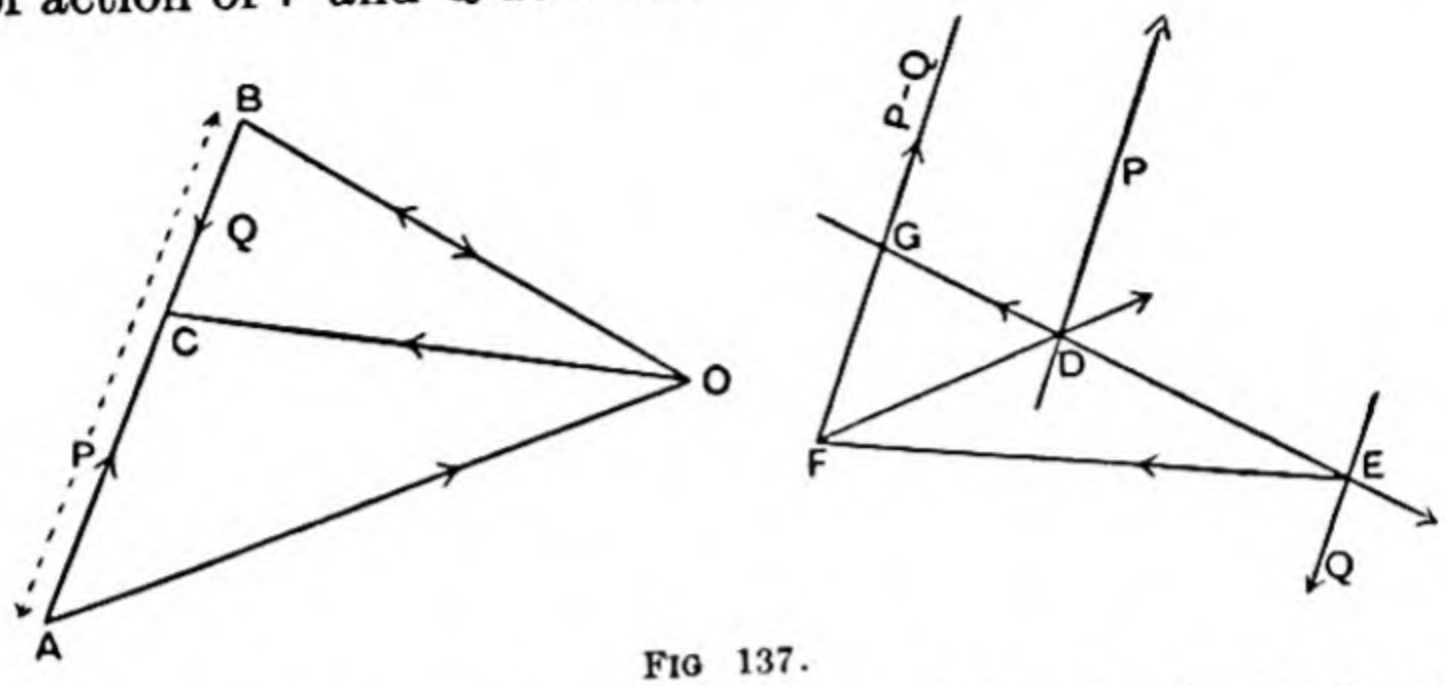



FIG 137.

*Note that* BO *is the side common to both triangles* CBO *and* ABO.

The construction and proof proceed as in the previous case, and we find that the resultant P – Q acts at a point G in ED, outside the greater force P, such that

$$P \,.\, DG = Q \,.\, EG.$$

The position of the point G in the line DE depends entirely on the magnitudes of P and Q, if D and E are fixed, so that if both forces are turned through the same angle the position of the point G is unaltered. This point is called the *Centre* of the Parallel Forces.

**Vertical Reactions by the Link Polygon.**

To find the vertical reactions X and Y, which are in equilibrium with the three vertical loads 15, 20, 10 lbs.

Instead of lettering the lines of action of the forces, letter the *spaces* between them, A, B, etc., and denote by *ab* the force between the spaces A and B. This is Bow's method of notation (see p. 174).

Draw the force diagram *ab*, *bc*, *cd* for the given forces, and take any point O, called the *pole*.

Join $a$, $b$, $c$, $d$ to O.

At any point $a$ on the line (AB) draw $ax$ parallel to $aO$ and $a\beta$ parallel to $Ob$. (Note that $Ob$ is the line common to the triangles $aOb$, $bOc$.)

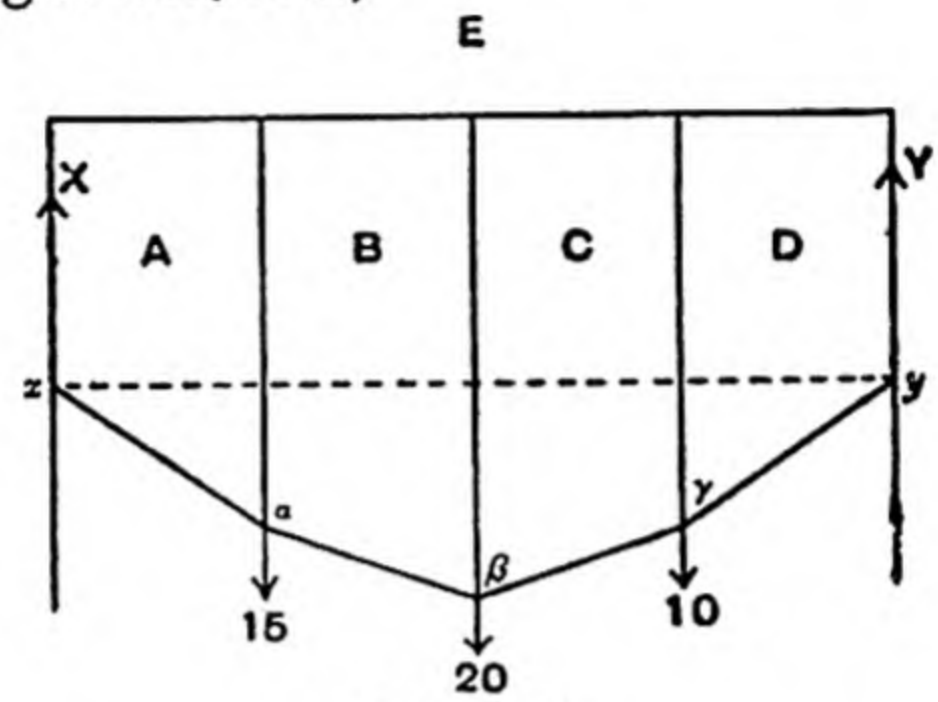



FIG. 138 (*a*).

Through $\beta$ draw $\beta\gamma$ parallel to O$c$.

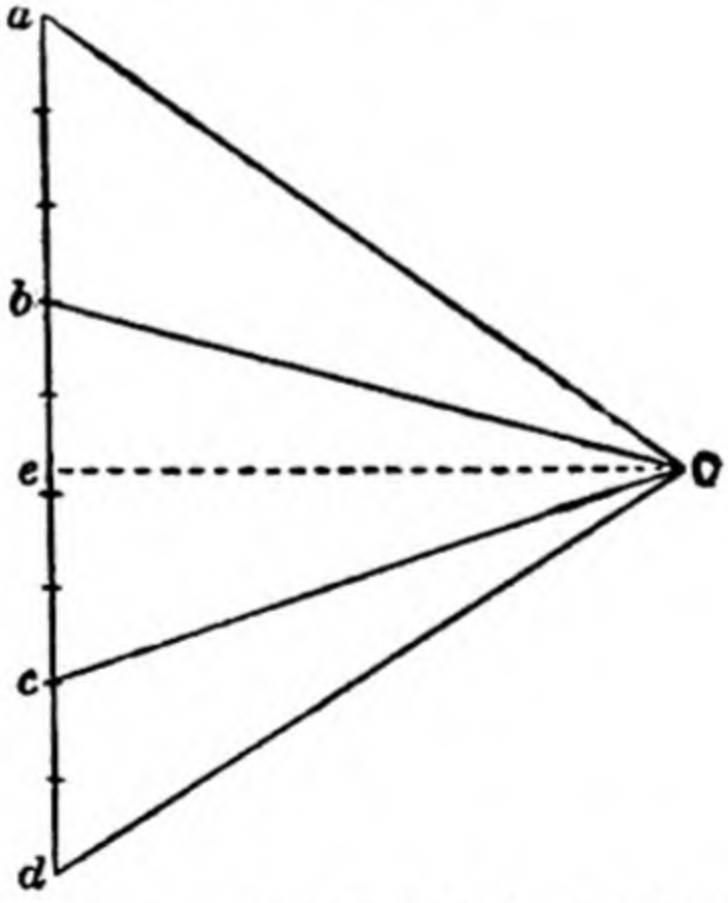



FIG. 138 (*b*). Scale 1" ≡ 10 lbs. (Reduced).

Through $\gamma$ draw $\gamma y$ parallel to O$d$.

Now, from the triangle of forces $abO$, $ab$ (15 lbs.) is equivalent to the forces $aO$, $Ob$.

Similarly, $bc$ (20 lbs.) is equivalent to $b$O, O$c$, and $cd$ is equivalent to $c$O, O$d$.

Hence the three given forces are equivalent to $a$O and O$d$.

Let the lines representing these forces cut the lines of action of X and Y at $x$ and $y$ respectively.

Join $xy$. Draw O$e$ parallel to $xy$.

Now $a$O acting at $x$ is equivalent to $ae$ and $e$O, from the triangle of forces $aeO$.

Similarly, O$d$ at $y$ is equivalent to O$e$ and $ed$.

Hence the given forces are equivalent to

$ae$, vertically downwards at $x$ and $e$O along $xy$;

$ed$, vertically downwards at $y$ and O$e$ along $yx$.

Therefore the given forces are equivalent to

$ae$ at $x$ and $ed$ at $y$;

$\therefore$ the vertical reactions are X $= ea$ and Y $= de$.

## EXAMPLES XXIII.

**1.** Determine graphically the line of action of the resultant of the given parallel forces:

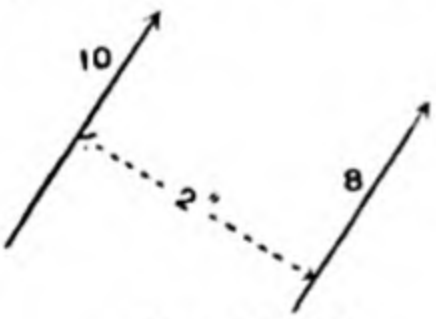



FIG. 139 (*a*).

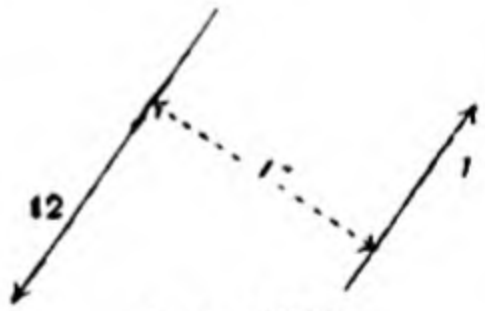



FIG. 139 (*b*).

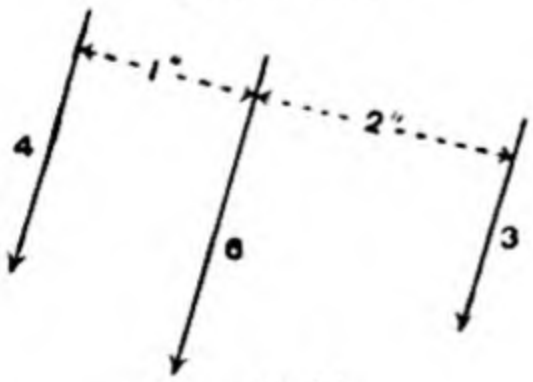



FIG. 139 (*c*).

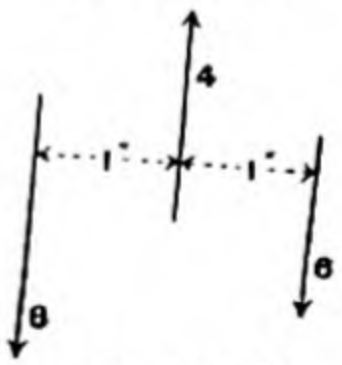



FIG. 139 (*d*).

Verify in each case by taking moments.

**2.** A horizontal bar 12 ft. long is supported at each end. If its weight may be neglected in comparison with a ton load which is placed upon it 3 ft. from one end, find graphically the pressure at each end.

**3.** A uniform beam 20 ft. long weighing 5 cwt. carries a load of 4 cwt. per foot run for 8 ft. of its length, beginning 3 ft. from one end. Find graphically the pressure produced on supports at its ends.

**4.** A roof formed by a triangular framework ABC, whose span AC = 12 ft., AB = 10 ft., BC = 8 ft., is loaded with 5 cwt. at B and also at the mid points of BA and BC. Find graphically the pressure on the supports at A and C.

**5.** A beam ABC, 30 ft. long, rests on vertical supports at A and at B, 20 ft. from A. It carries a load of 3 tons at C and 10 tons at D, the mid point of AC. Find graphically the reactions at A and B. (Draw the lines representing X and Y, the reactions of the supports, *above* AB, and draw the lines representing the loads *below* AB. Then letter the spaces.)

**6.** A light framework is supported at its ends A and B, which are 12 ft. apart. The framework carries a load of 10 cwt. 3 ft. from A and one of 5 cwt. 9 ft. from A; find graphically the vertical reactions at the supports.

**7.** A girder AB, 50 ft. long, is supported at C 18 ft. from A, and at D 12 ft. from B. It is loaded with 10 tons at A, 15 tons at B and 20 tons at its mid point. Find graphically the reactions at C and D.

**8.** Find graphically the forces X and Y which are in equilibrium with the three given forces.

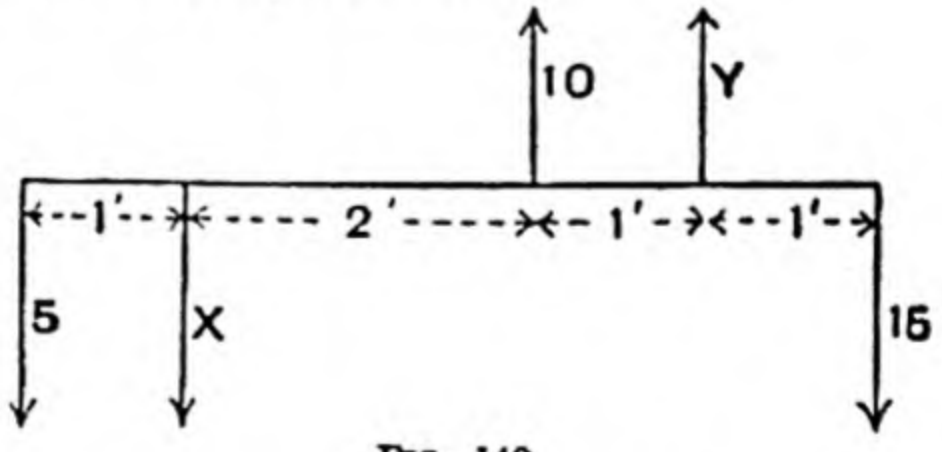



FIG. 140.

**9.** The centre of gravity of a number of particles, whose total weight is W, is at G. One of the particles, whose weight is $w$, is moved a distance $a$. Prove that the C. of G. of the

system moves a distance $\frac{aw}{W}$ in a direction parallel to the motion of $w$.

A weight is displaced a distance of 50 ft. across the deck of a ship which weighs 6000 tons. What must the weight be if its displacement shifts the C. of G. of the ship a distance of one inch ?

**10.** A uniform bar 1 foot long weighing 1 lb. has a sphere of iron fastened to each end, one of 3-inch radius and weight 10 lbs., the other of 2 inches radius and weight 3 lbs. Find by a graphical construction the C. of G. of the body.

**11.** A horizontal light rod ABCD carries a load of 30 lbs. at B and 12 lbs. at C. It rests on two supports, one of which is at A where the reaction is 10 lbs. Find graphically the position of the other support, given AB = 3 ft., BC = $5\frac{1}{2}$ ft., CD = 1 ft.

**12.** A heavy uniform beam 8 ft. long weighs 20 lbs., and at one end it carries a load of 20 lbs., and at the other end a load of 10 lbs. The beam thus loaded is to be carried on a certain support. Find graphically the position of the support.

If the beam is to rest on two supports which are to divide the load in a given ratio, show how to find on your diagram any number of suitable pairs of points, and find the position of the supports when they are at a distance 4 ft. apart, the pressures on them being in the ratio 3 : 2.

**Graphical Representation of a Moment. Moment of the Resultant.**

The moment of a force P about a point O is measured by the product $Pp$, where $p$ is the perpendicular from O to the line of action of the force.

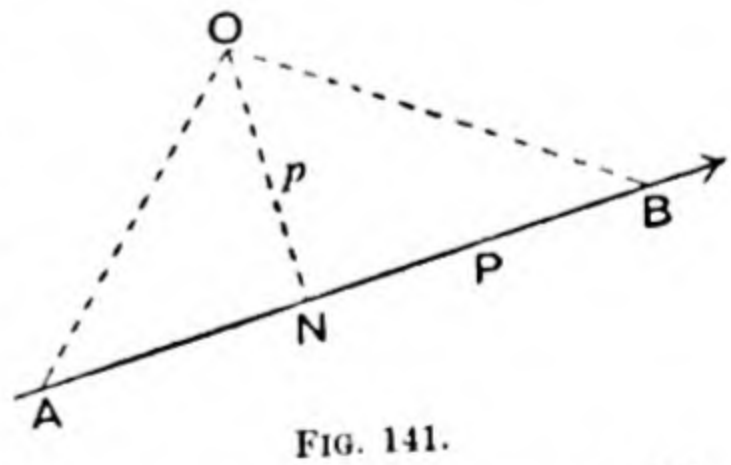


FIG. 141.

If a length AB is marked off along the line of action of P, to represent P in magnitude, this moment is represented by $AB \times p$. The area of the triangle is $\frac{p \times AB}{2}$;

∴ twice the area of the triangle AOB represents the moment of the force.

If P is 5 lbs. weight, we may draw AB 5″, on the scale 1″ ≡ 1 lb.

If ON is 2 ft., we may draw ON 2″, on the scale 1 ft. ≡ 1″.

Twice the area of the △ will be 10 sq. in., which represents the moment of the force (10 lbs.-ft.).

**Moment of the Resultant.**

It has been shown in Part I. p. 134, that for a system of parallel forces, the moment of the resultant about any point equals the algebraical sum of the moments of the parallel forces about that point.

We shall now show that the same theorem holds good for forces which are not parallel.

Let the forces P, and Q acting at A, be represented by

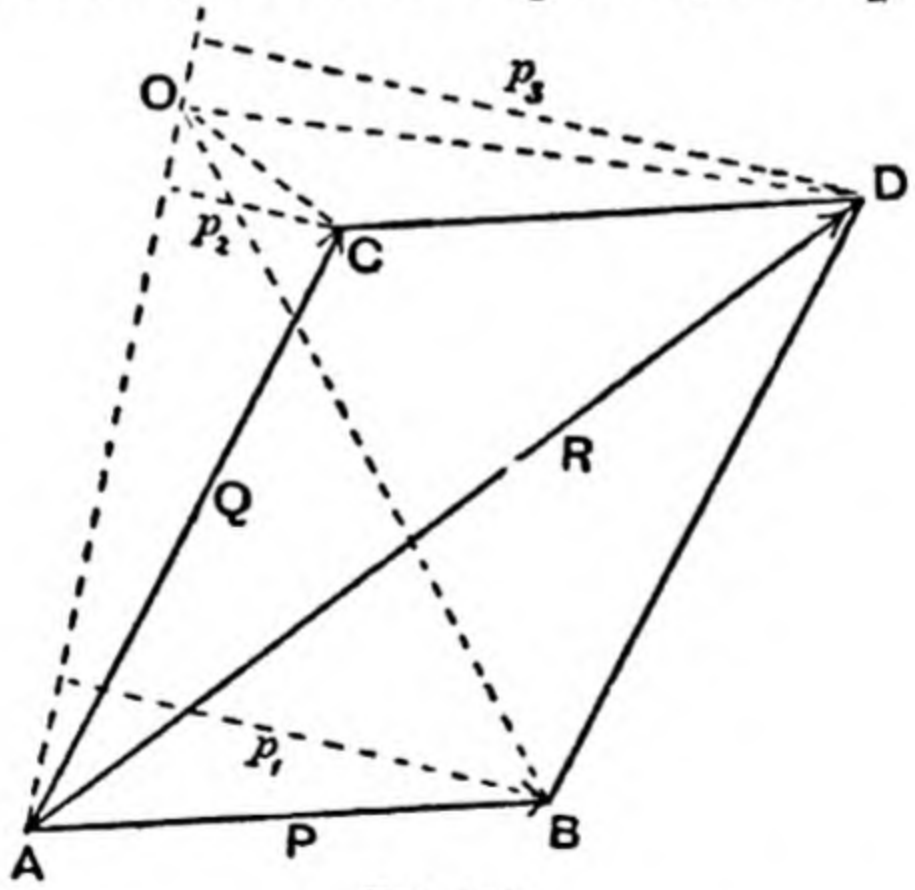



FIG. 142.

the sides of the parallelogram ABDC, so that AD represents their resultant. Take moments about O.

Let the perpendiculars from B, C, D on the line AO be $p_1$, $p_2$, $p_3$ respectively.

The algebraical sum of the moments of P and Q about O

$$= 2\overline{AOB} + 2\overline{AOC}$$
$$= OA \cdot p_1 + OA \cdot p_2$$
$$= OA(p_1 + p_2) = OAp_3 = 2\overline{AOD}$$
$$= \text{moment of R about O.}$$

The geometrical fact that $p_1 + p_2 = p_3$ may be seen by drawing CL perpendicular to DT.

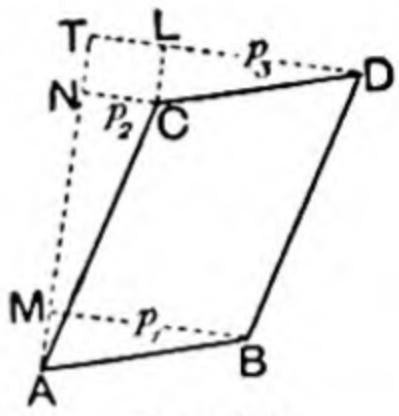


FIG. 143.

Since ABM and CDL are congruent, $DL = p_1$ and

$$DT = DL + LT.$$

Prove the theorem for the case in which O lies within the angle CAB.

**Alternative Proof.**

Through O draw OC parallel to the direction of P, cutting the line of action of Q at C.

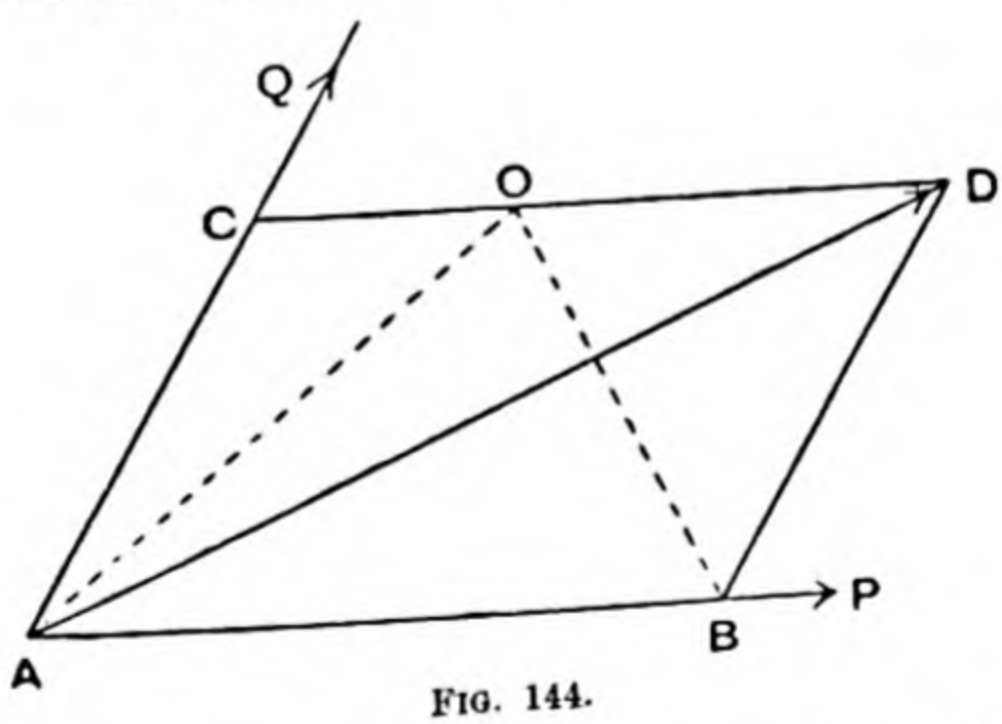


FIG. 144.

Take the length AC to represent the magnitude of Q, and *on the same scale* suppose AB to represent P. Complete the parallelogram ABDC; then on the same scale AD represents R.

Algebraical sum of the moments of P and Q about O

$$= 2\overline{AOB} - 2\overline{AOC}$$
$$= 2\overline{ADB} - 2\overline{AOC}$$
$$= 2\overline{ADC} - 2\overline{AOC}$$
$$= 2\overline{AOD}$$
$$=$$ moment of R about O.

The theorem may now be generalised for the case in which any number of forces in one plane, acting on a rigid body, have a resultant.

Let the forces be $p$, $q$, $r$, ... , and let O be the point about which moments are taken.

Let $p$ and $q$ meet at A; then the algebraic sum of the moments of $p$ and $q$ about O equals the moment of their resultant $R_1$ about O.

Let $R_1$ meet the force $r$ at B; then the sum of the moments of $R_1$ and $r$ about O equals the moment of their resultant $R_2$ about O, *i.e.* the algebraic sum of the moments of $p$, $q$, $r$ about O = moment of their resultant $R_2$ about O.

Similarly for any additional number of forces.

EXAMPLE. Three forces of 5 lbs., 10 lbs., 7 lbs. act on a rigid body so that their lines of action form an equilateral triangle of side 1 foot. To find the line of action of their resultant.

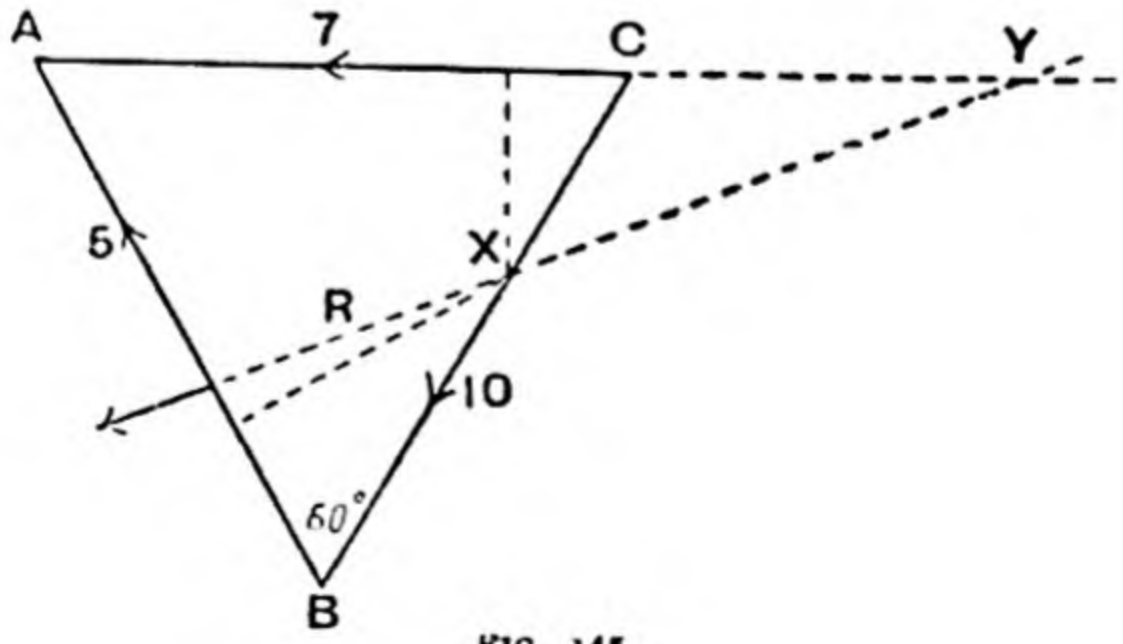



FIG. 145.

Suppose the line of action of the resultant to cut BC at X, a distance of $x$ ft. from B.

The algebraical sum of the moments of the three given forces about X = moment of their resultant about X, = 0 since the resultant goes through X; *i.e.*

$$5x \sin 60 - 7(1-x) \sin 60 = 0,$$

or $$x = \tfrac{7}{12} \text{ ft.}$$

Similarly, if the resultant cut AC at Y, $y$ feet from A, we get $y = 2$. YX is the line of action of the resultant.

The magnitude of R may now be found by taking moments about C. (Measure the length of the perpendicular from C to the line of action of the resultant.)

## EXAMPLES XXIV.

**1.** A rigid body is acted upon by three forces, P, Q, R, whose lines of action form an equilateral triangle ABC of side 2 ft. If P along AB = 5 lbs., Q along BC = 8 lbs. and R along AC = 6 lbs., find the line of action of their resultant.

**2.** Forces of 2, 3, 5, 6 lbs. wt. act in order along the sides of a square whose side is 1 ft. Find the line of action of their resultant.

**3.** Four forces each of 1 lb. wt. act along AB, CB, DC, AD, the sides of a square ABCD. Find the direction and magnitude of their resultant.

**4.** ABC is an isosceles triangle. AB = AC = 3 ft. BC = 2 ft. P, 5 lbs., acts along AD, drawn at right angles to BC; Q, 4 lbs., acts along BC, and R, 6 lbs., acts along CA. Find the line of action and magnitude of the resultant.

**5.** Find the line of action of the equilibrant of the three given forces acting at X, Y and Z. (Draw the isosceles triangle ABC. AB = 6 cms., AC = CB = 5 cms.)

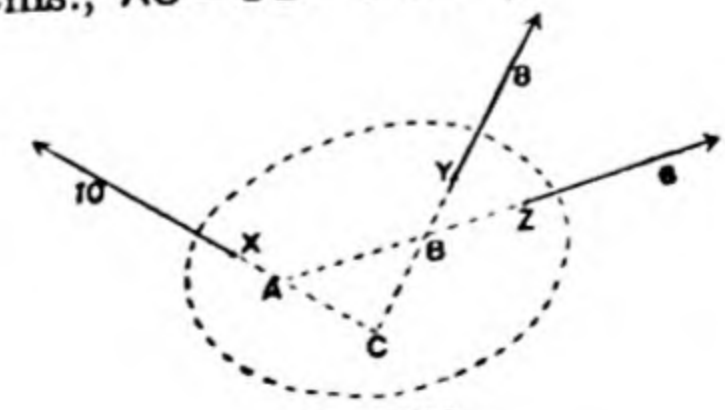



FIG. 146.

**6.** Draw a triangle ABC, AB = 9·7 cms., BC = 11·5 cms., CA = 6·3 cms. If a force of 8 lbs. acts along AB, 13 lbs. along BC and P lbs. along CA, find the magnitude of P if the resultant cuts BC produced at a point 10 cms. from C.

**7.** Find the magnitude of the forces P, Q, R, which, acting along the sides of the triangle ABC, have the same effect as the given force of 10 lbs.

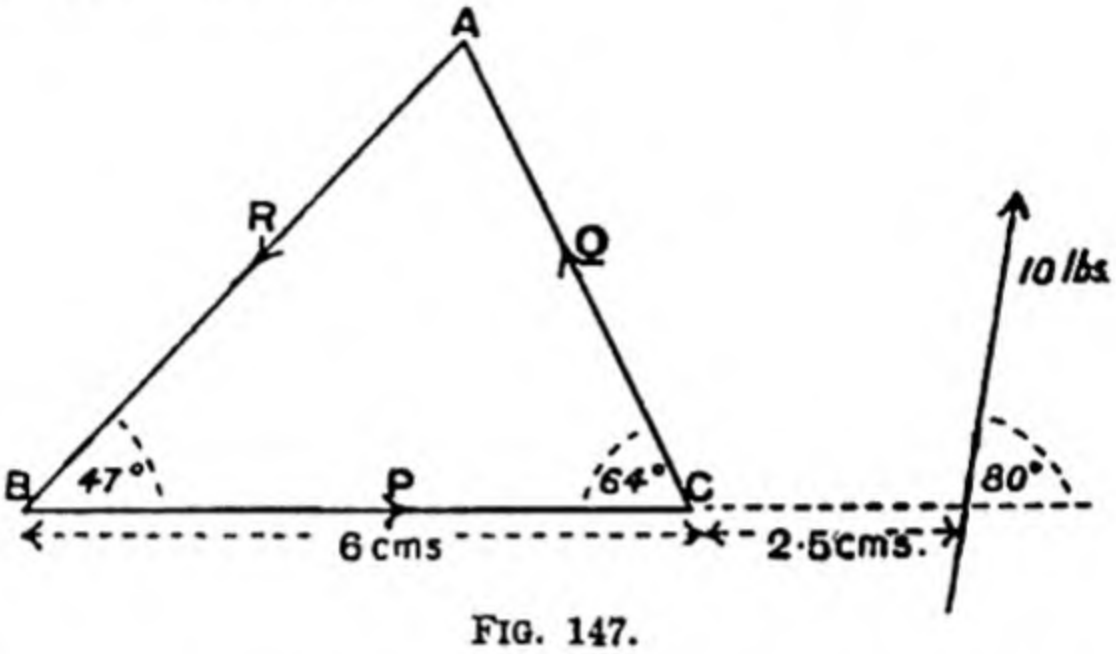



FIG. 147.

**8.** A square board ABCD, lying on a smooth horizontal table, can turn freely about a pin at O, the mid point of AB. A force of 12 lbs. acts upon it along AD, and one of 4 lbs. from B to C. Find the magnitude of a third force along the line CD which will keep the board at rest, and find the direction and magnitude of the force exerted by the pin.

**9.** Forces equal to the weights of 3 lbs., 5 lbs., 7 lbs. act along three of the sides of a square taken in order: find the magnitude of the force in the remaining side, which, together with some force acting through the centre of the square, will balance the three given forces.

**10.** If six forces, of relative magnitude 1, 2, 3, 4, 5 and 6, act along the sides of a regular hexagon taken in order, show that the single equivalent force is of relative magnitude 6, and that it acts along a line parallel to the force 5, at a distance from the centre of the hexagon $3\frac{1}{2}$ times the distance of a side from the centre.

## CHAPTER VII.

### FRAMEWORKS. BOW'S NOTATION.

We know from experience that if a bar is supported at its two ends, it is likely to break if a heavy weight is attached to it between the points of support, but the same bar will resist much greater forces if they are applied at the ends and act along the rod.

If these forces tend to compress the bar, they are called thrusts, and the bar is a strut. If the forces tend to stretch the bar, they are tensions, and the bar is a tie. In a diagram, bars which are compressed are marked +, and those which are stretched are marked −. Another method which is in general use is to indicate a *strut* by double lines, a *tie* by a single line.

Engineers find that hollow rods of circular section or rods having a cross section thus, H, are most suitable for struts, whereas ties are usually made solid and of circular or rectangular section.

A system of bars united by free joints is called a "frame-work." "Trusses" used by engineers are practically frameworks, except that a single bar or member is often continuous through a joint.

In frameworks made to support a roof, the rods are generally connected by a pin passing through holes made at

their extremities. We shall suppose the external forces which act on the frame to be applied to these pins, so that approximately the forces are acting at a point. Any stiffness at these joints will cause the framework to be more rigid, and the error in neglecting this will generally be on the side of safety.

Let AB, AC represent two bars jointed at A, at which point an external vertical load (W) is to be carried.

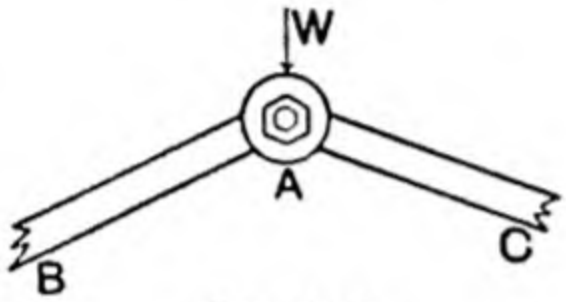



FIG. 148.

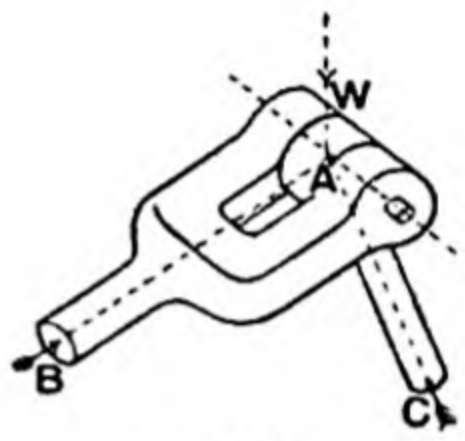



FIG. 149.

The thrust on the bar AB at A will be transmitted to it by the pin at A, and the bar will be kept in equilibrium by a thrust due to the pin at the other extremity B.

FIG. 150.

If these forces are so great that the weight of the bar may be neglected, they must be equal to one another and act along the line of the bar.

The thrust of the bar AB *on the pin* at A will therefore be in the direction BA.

Similarly, the thrust Q of the bar CA on the pin at A will be along CA, so that the forces on the pin are equivalent to three forces W, P and Q acting approximately at a point (Fig. 149).

The forces acting on the bars produce a state of strain in the material of the bar. If part of the bar to the left of DE is removed, a force P acting to the left would have to be introduced to keep the part on the right in equilibrium. This force P measures the tensile stress in the bar.

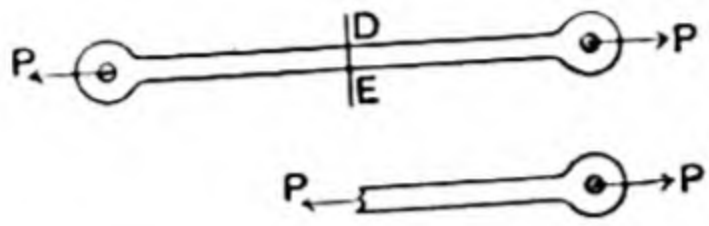



FIG. 151.

A strut may also be considered in the same way.

The loads carried by a bar at points where it is not supported tend to break and to bend the bar. This will be considered in Chap. VIII.

A framework used as a beam is called a *Girder*.

When the load to be carried is distributed along a girder, we may, when calculating the thrusts produced in the

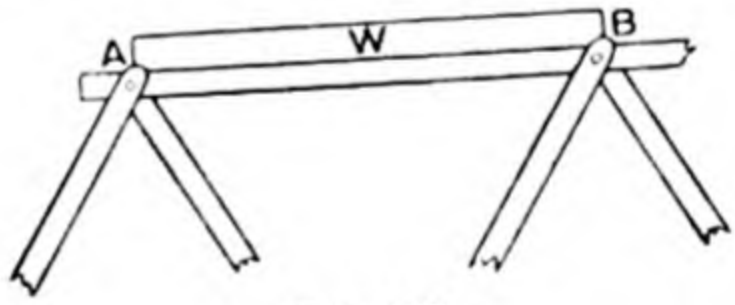



FIG. 152.

struts, consider that a load W, uniformly distributed along the girder AB, is equivalent to a vertical downward thrust of $\frac{W}{2}$ at A and $\frac{W}{2}$ at B.

It may happen that the weight of the framework itself is a considerable addition to the dead load carried. In such cases this weight is added to the external loads and distributed at the joints.

Consider the case of the king-post truss in Fig. 153.

Suppose the weight of the roof carried by the truss to be $w_1$.

Each rafter AD, DC, CE, EB may be considered as carrying a load of $\frac{w_1}{4}$. This is equivalent to $\frac{w_1}{8}$ at the extremity of each rafter.

The load at the joints will therefore be $\frac{w_1}{8}$ at A and B, $\frac{w_1}{4}$ at D, C and E.

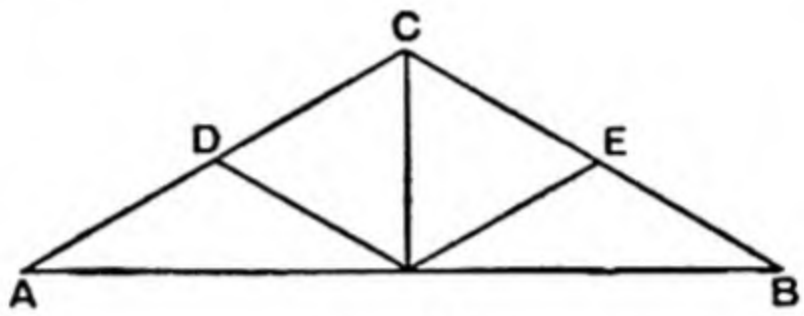



FIG. 153.

If the weight of the truss itself is considerable and equal to $w_2$, the total load $w_1 + w_2 = W$ is considered to be divided up equally between the same rafters, so that the loads at the joints will be $\frac{W}{8}$ at A and B, $\frac{W}{4}$ at D, C and E.

The solution of problems on frameworks is much facilitated by the adoption of Bow's method of notation (see Step 3, p. 175).

EXAMPLE 1. The given triangular framework XYZ of light jointed bars is supported by vertical forces P and Q, and is subject to a vertical pressure of 2 cwt. at its apex. XY = 8 ft., YZ = 10 ft., ZX = 6 ft.

The weights of the bars are considered to be negligible in comparison with the external forces and the thrusts in the bars.

The whole framework may be looked upon as a rigid body acted upon by three external parallel forces which are in equilibrium.

*Step* 1. The forces P and Q are generally found by the graphical method given in Ch. VI. p. 162. They may also be calculated by taking moments about Y or Z. From measurements on the figure, we have $Q \cdot 10 = 2(6{\cdot}45)$; $\therefore Q = 1{\cdot}29$ cwt.

Since $P + Q = 2$, $P = {\cdot}71$ cwt.

*Step* 2. Draw the lines of action of the external forces, *which must be kept outside the framework.*

*Step* 3. The lines representing the external forces and the lines denoting the bars divide the plane of the paper into various spaces. Label these spaces with capital letters A, B, C, D.

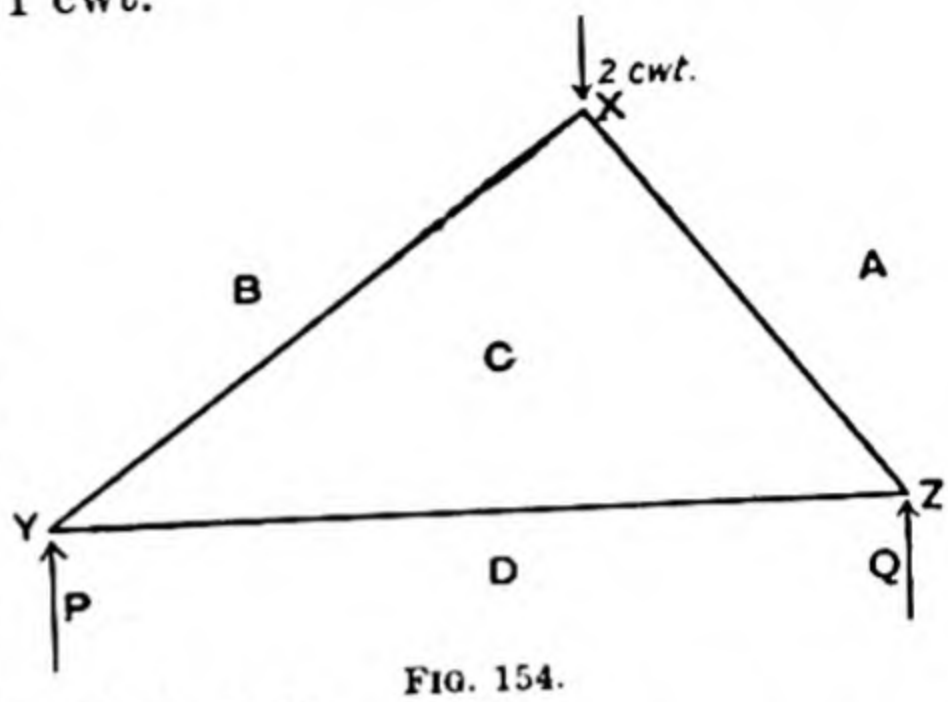
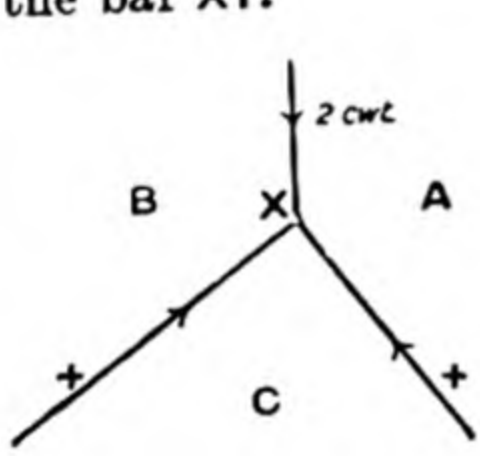
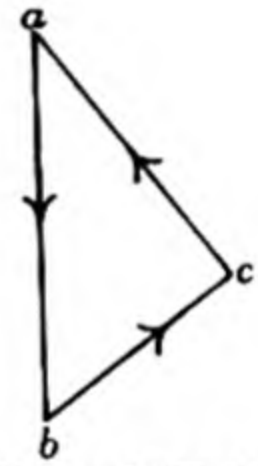



FIG. 154.

The space B extends indefinitely to the left, and is separated from the other spaces by the line of action of the weight, the force P and the bar XY.




Space diagram.



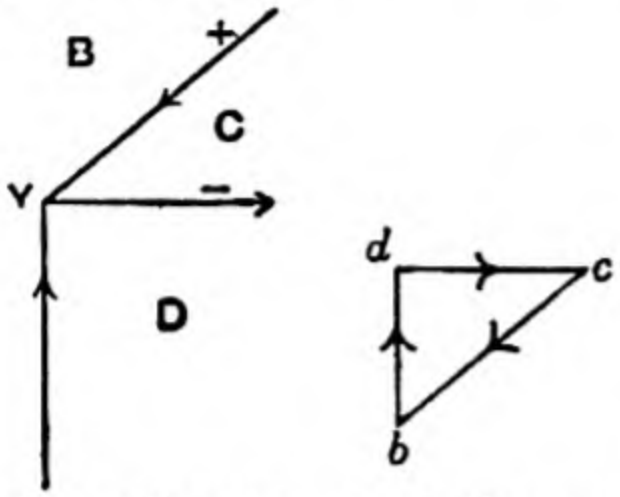

Force diagram.

FIG. 155 (*a*).

(*a*) Draw the triangle of forces *abc* for the 3 forces acting at X. The force between the spaces A and B is called *ab*, between B and C, *bc*, etc. It appears from the force diagram that the forces *bc*, *ca* both act towards X. Both bars are therefore pushing at X, and are struts. They are now marked + in the space diagram.

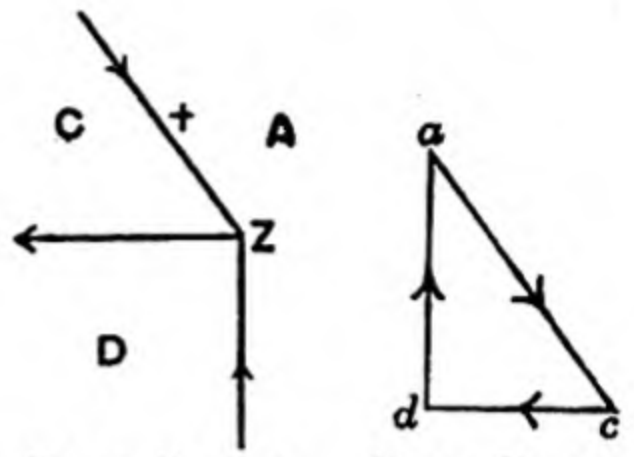
($\beta$) Draw the triangle of forces for the joint Y. At X the thrust in the bar YX is *bc*, but at Y the thrust is *cb*. It appears from the triangle of forces that the force at Y due to

the rod YZ is *dc*, so that YZ is a tie, and is marked – in the space diagram.




Space diagram. Force diagram.

FIG. 155 (*b*).




Space diagram. Force diagram.

FIG. 155 (*c*).

($\gamma$) Similarly for the forces at Z. *ac* acts towards Z and *cd* away from Z.

The last three stages ($\alpha$, $\beta$, $\gamma$) in practice are all combined into one as follows :

*Step* 4. Draw a force diagram to represent all the external forces. Since these are parallel, we shall have a straight line. Scale $2'' \equiv 1$ cwt. *ab* represents 2 cwt., *bd* represents P and *da* represents Q.

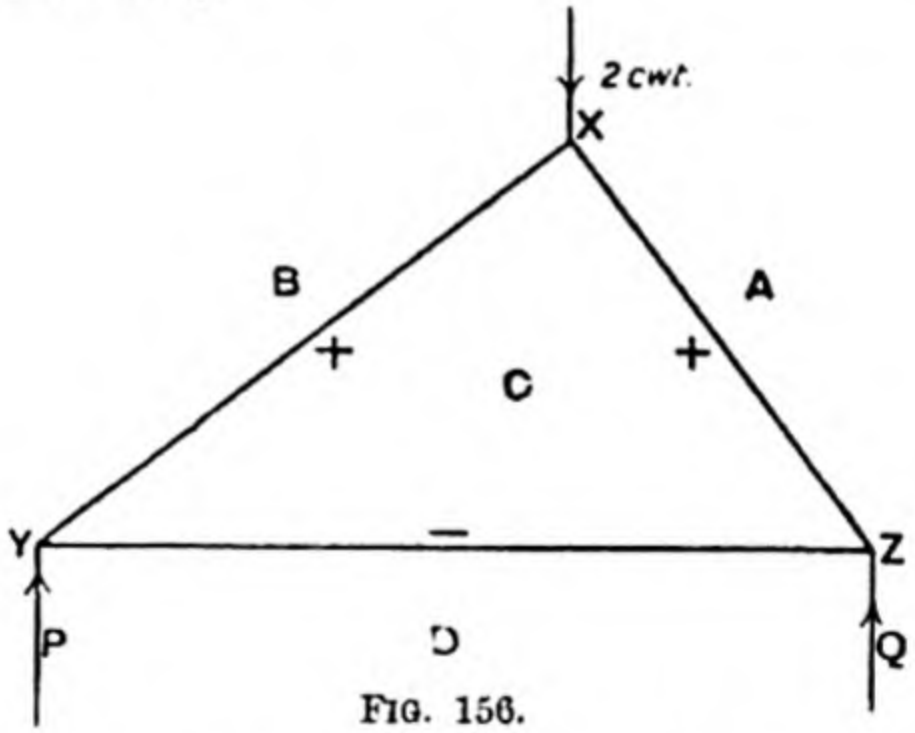



FIG. 156.

Draw the triangle of forces *bdc* for the forces acting at Y. They are *bd* upwards, *dc* horizontal (it must be to the right, as *cb* is to be parallel to XY).

*Note.*—*bd* upwards, not *db*, since *db* is downwards in the force diagram.

The rod YZ is now marked –, since it is pulling at Y, and is

therefore a tie, and YX is marked +, since the direction of its thrust is *cb* at Y, and it is therefore a strut.

Now consider the forces at X. They are *ab*, *bc* (since the rod YX is pushing at X), so that *ca*, which completes the triangle of forces *abc*, represents the thrust of the rod XZ, which is now marked + in the space diagram.

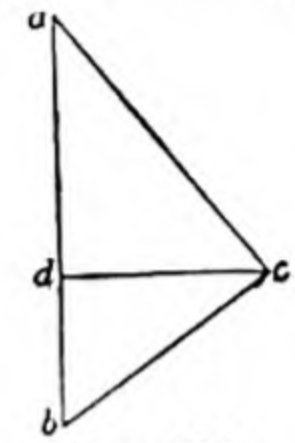



FIG. 157.
• Force diagram.
Scale 2" ≡ 1 cwt.

*N.B.*—Arrows are not drawn on the force diagrams.

It will be seen that the triangle of forces *adc* for the point Z is already drawn, and the fact that the lines representing the forces at Z are parallel to their directions in the space diagram is a check on the accuracy of the drawing.

By measurement, $bc = 2{\cdot}4''$, $cd = 1{\cdot}9''$, $ca = 3{\cdot}2''$; ∴ thrust in YX = 1·2 cwt., thrust in XZ = 1·6 cwt., tension in YZ = ·95 cwt.

The Space Diagram and Force Diagram are sometimes called "Reciprocal" figures.

EXAMPLE 2. The diagram represents part of a Warren girder, in which the bars form equilateral triangles. It is supported by two vertical forces P and Q, and carries loads of 10 cwt. and 5 cwt. at the joints of the upper boom.

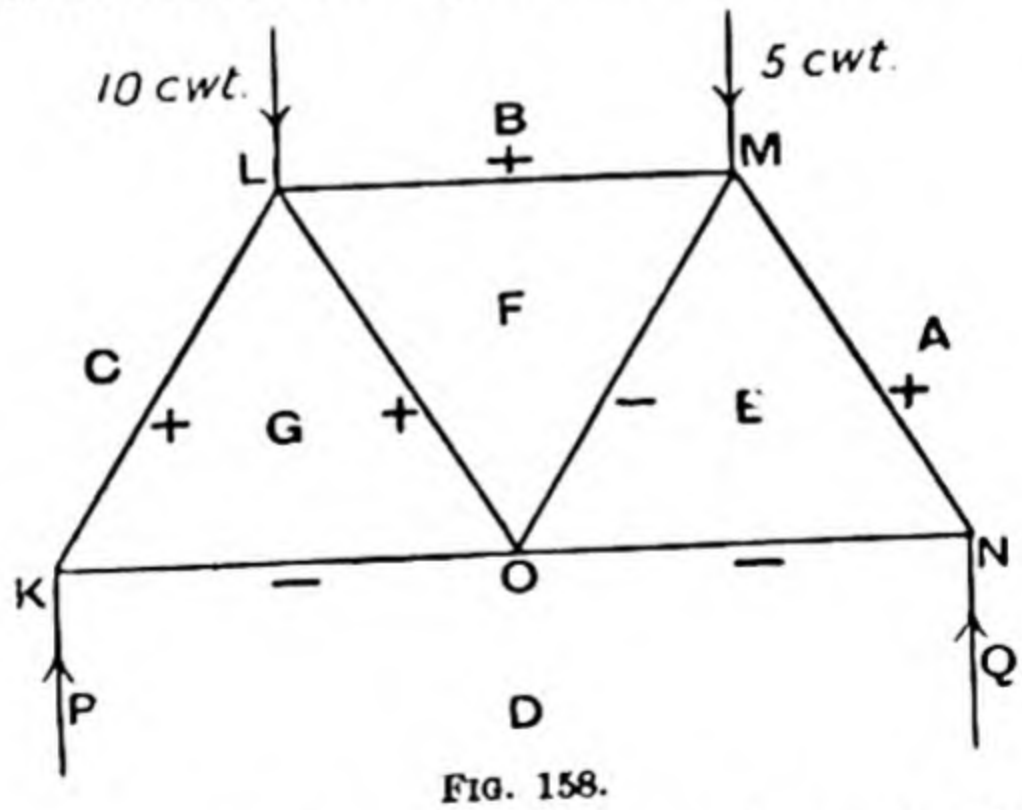



FIG. 158.

• The force diagrams are reduced in size. They should be drawn *at least* as large as is indicated by the scale.

1. Take moments about K for the external forces. If each side of an equilateral triangle is 2 units, we have

$$10 \cdot 1 + 5 \cdot 3 = Q \cdot 4 ; \quad \therefore Q = 6\tfrac{1}{4} \text{ cwt.} ;$$

and $\therefore P = 8\tfrac{3}{4}$ cwt., *or* determine these reactions by a graphical construction.

2. Draw the lines of action of the external forces *outside* the frame.

3. Letter the spaces.

4. The force diagram for the external forces is the line

$$ab = 5 \text{ cwt.}, \quad bc = 10 \text{ cwt.}, \quad cd = P, \quad da = Q.$$

Draw the triangle of forces *cd*, *dg*, *gc* for the forces at K.

*N.B.—It is always necessary to begin at a point where three forces act.*

Note that *cd* represents P (not *dc*).

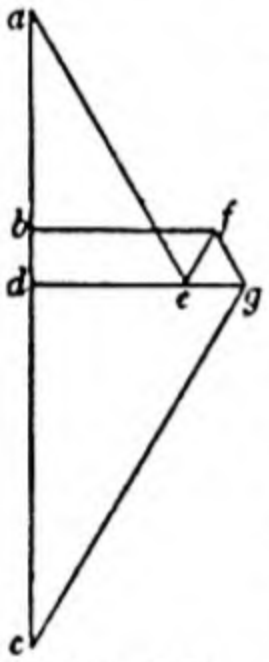



FIG. 159.
Force diagram.
Scale 2″ ≡ 5 cwt.

*dg* must act to the right, so that *gc* may be parallel to the bar LK; $\therefore$ *dg* is a tension, and KO is marked −, *gc* is a thrust.

Now draw the polygon of forces for the four forces acting at L.

In the force diagram we already have *bc*, *cg*. Draw *gf* parallel to OL; then *fb* must be drawn parallel to ML, so that *f* is fixed and the polygon is *bcgfb*.

Then draw the polygon of forces *fgdef* for the forces at O. Note that *e* must be a point in *dg*, so that *ef* is parallel to the rod MO.

The forces at M are represented by the polygon *abfea*, so that when we join *ea* the line should be parallel to the rod MN. The polygon of forces for N is now already drawn, and its correctness is a test of the accuracy of the drawing.

By measurement, we have in cwts.

$$cg = +10{\cdot}1, \quad dg = -5{\cdot}05, \quad fb = +4{\cdot}3, \quad fe = -1{\cdot}45,$$
$$fg = +1{\cdot}45, \quad ae = +7{\cdot}2, \quad ed = -3{\cdot}6.$$

In practice KON would be a single rod strong enough to bear a tension of 5·05 cwt.

## EXAMPLES XXV.

**Roof Trusses and Frameworks.**

**1.** Triangular framework. BC = 12 ft., BA = 10 ft., AC = 8 ft. Five tons at vertex A, vertical reactions at B and C, BC being below A and horizontal.

**2.** Simple "king-post" frame for span up to 20 ft. BC = 20 ft., AD = 4 ft. 5 cwt. at A. It will be found that there is no additional stress in the king-post AD, which is inserted to support BC.

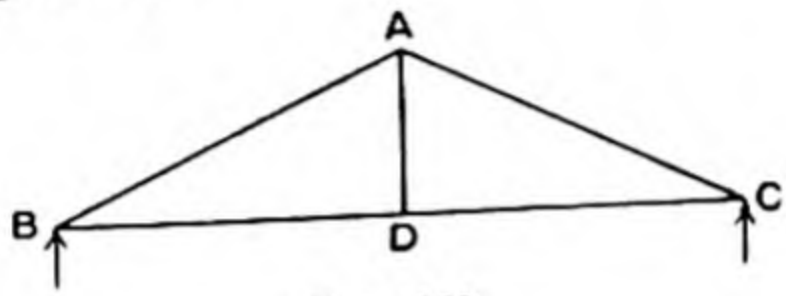



FIG. 160.

**3.** Same frame with 4 cwt. distributed uniformly along AB and AC. This is equivalent to 2 cwt. at A and B for rod AB, and 2 cwt. at A and C for rod AC.

**4.** King-post frame, with struts, for spans up to 30 ft. BC = 30 ft., AD = 6 ft., AE = EB = AF = FC. 4 cwt. at A, E and F.

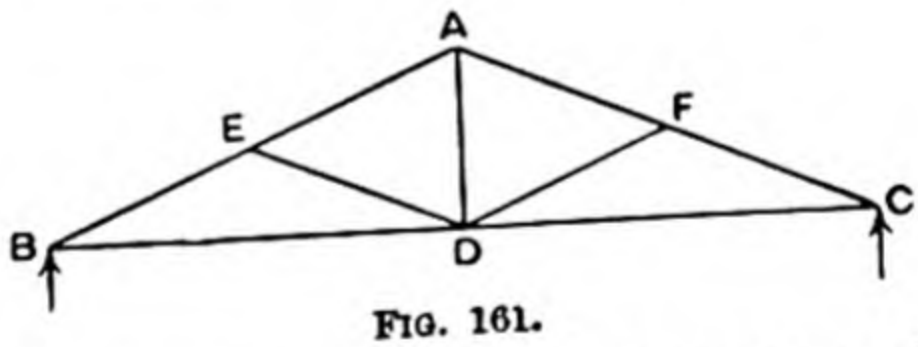



FIG. 161.

**5.** The same frame with 5 cwt. at E, 6 at A and 4 at F.

**6.** Swiss roof truss. BC = 20 ft., D 1 ft. above BC, AD = 4 ft. 8 cwt. at A.

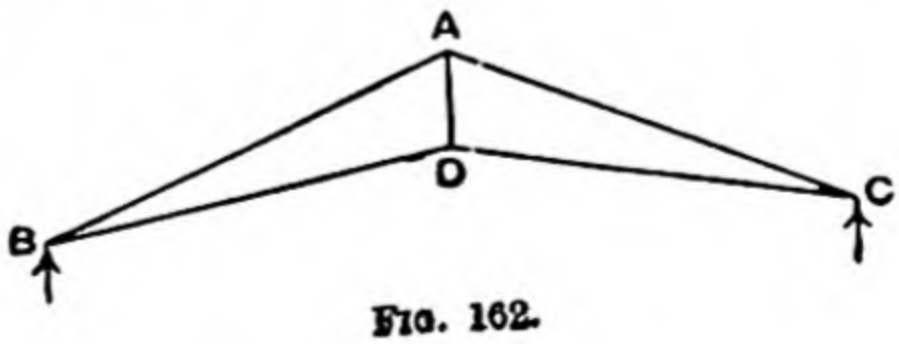



FIG. 162.

**7.** Compound Swiss truss. BC = 30 ft., AB = AC = 18 ft. FD = EG = 2 ft. 5 cwt. at F, A and G.

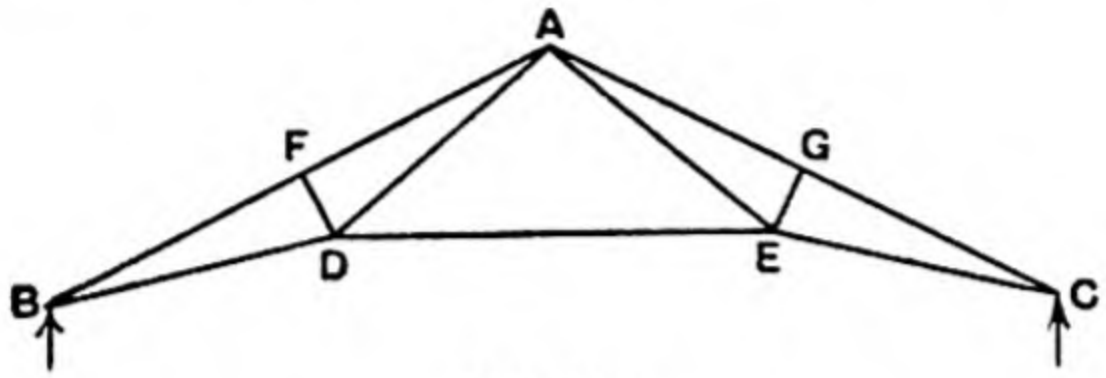



FIG. 163.

**8.** Same frame as Qu. 7. 6 cwt. at F, 5 cwt. at A and 4 at G.

**9.** Queen-post frame. BC = 25 ft. A 5 ft. above BC. FD, GE, the queen posts, perpendicular to BC at the mid point of AB and AC.

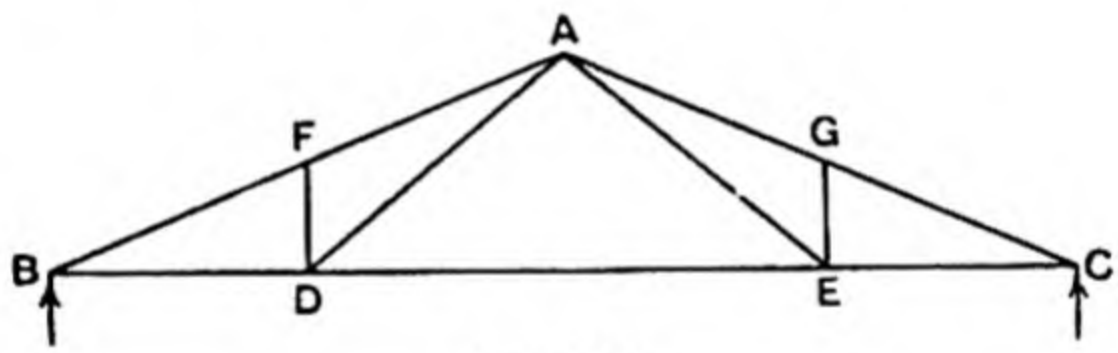



FIG. 164.

Loads of 1 ton at A, $\frac{1}{2}$ ton at F and G.

**10.** Same frame as Qu. 9. 1 ton at F, 1 ton at A, 2 tons at G.

**11.** Compound queen post. BC = 50 ft., AM = 10 ft. AB, AC are trisected. 500 lbs. at D and G, 1000 lbs. at E and F.

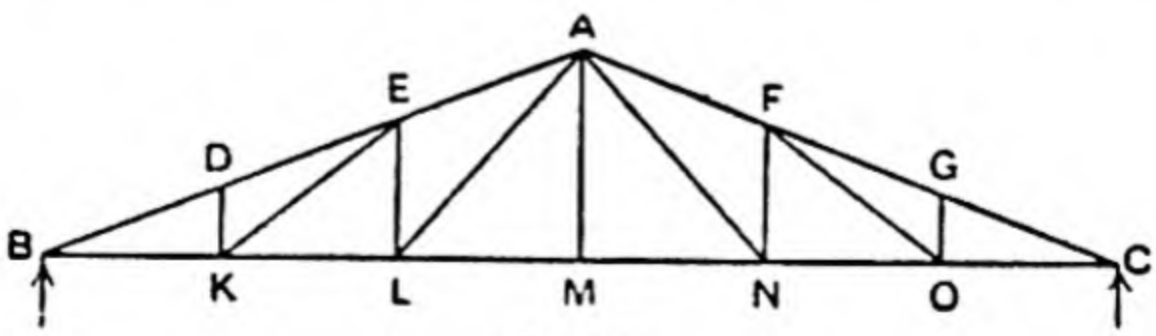



FIG. 165.

**12.** English truss. BC = 60 ft., AG = 10 ft. AB, AC are divided into four equal parts.

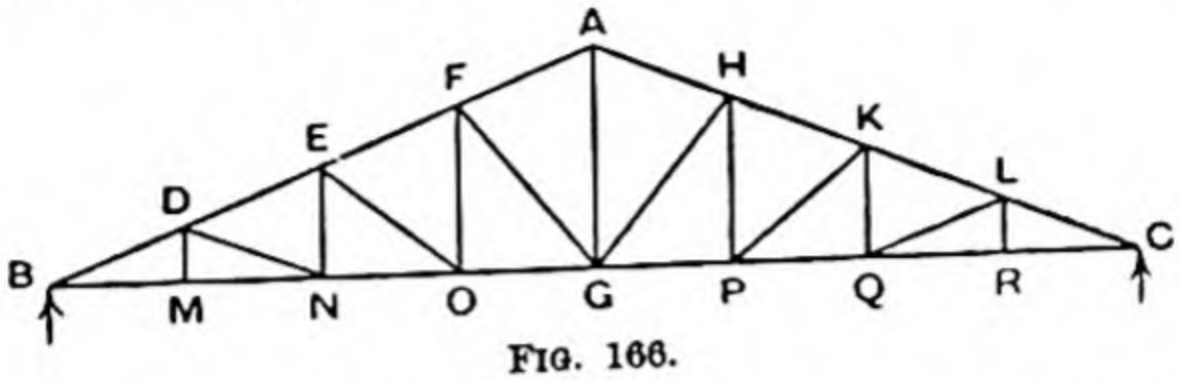



FIG. 166.

Loads at B and C, 600 lbs. At A and all other joints on AB and AC 1000 lbs. each.

**13.** Mansard roof. AB = 20 ft. Struts at the mid points of AC and CB.

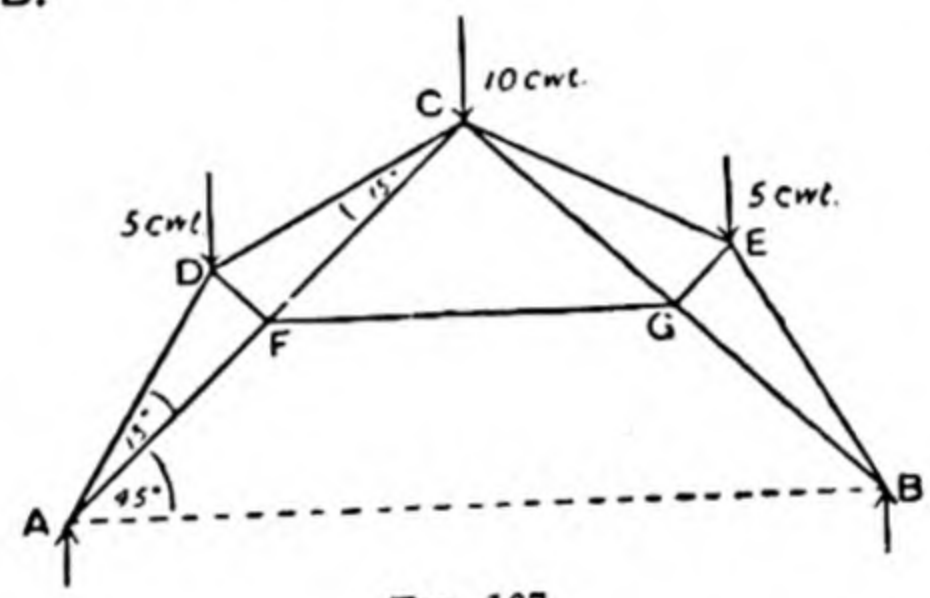



FIG. 167.

**14.** Framework smoothly jointed at A, B, C, D, E, F, G. Each rod is either horizontal or vertical or inclined at 45° to the vertical. Weights of 3, 4, 7 tons are supported at E, F and G,

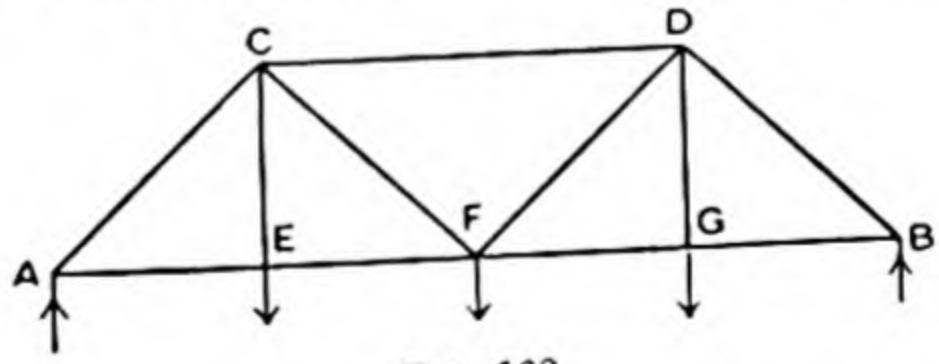



FIG. 168.

and the framework is supported at A and B. Find the stresses produced in the rods.

**15.** In the jointed framework DCF and ECF are equilateral triangles. ACE and BCD are straight, and the angles

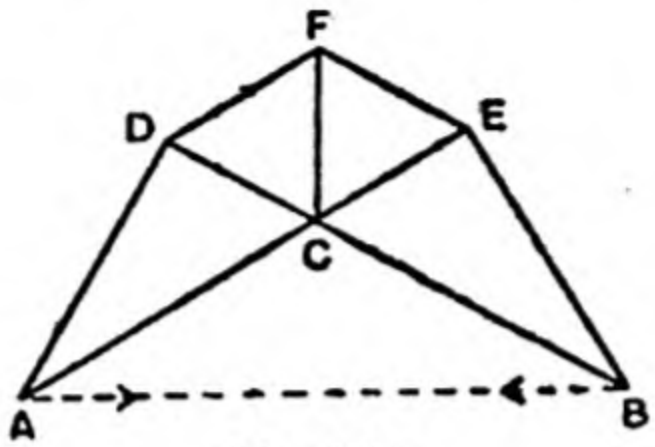



FIG. 169.

ADC and BEC are right angles. Equal forces of 5 cwt. are applied at A and B in the line AB. Draw a force diagram for the frame, and find the stresses.

**16.** The framework is made of light stiff rods freely jointed together and hinged at a fixed point O. OA = 12 ft., OB = 15 ft. C is one-third of the way up the diagonal of the rectangle

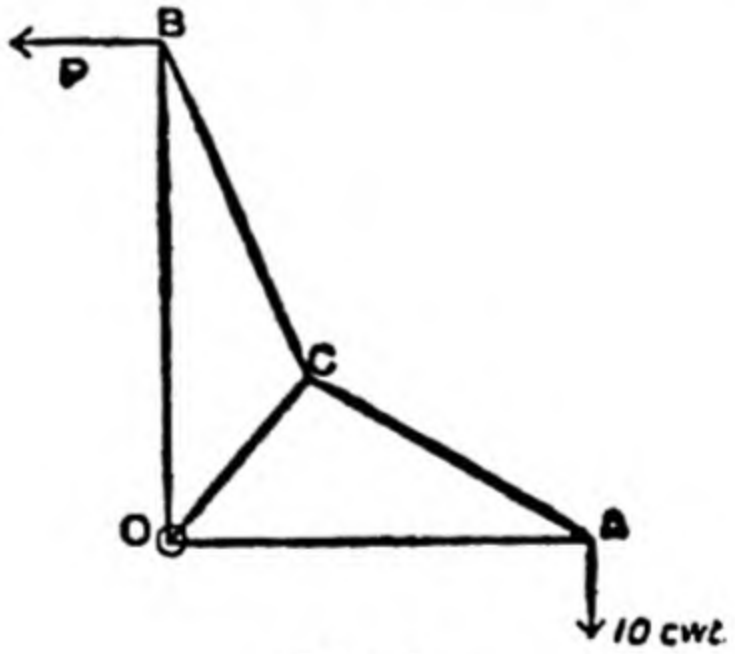



FIG. 170.

OA, OB. A weight of 10 cwt. hanging from A is supported by a horizontal force P applied at B. Determine the stresses in the three rods which meet at C.

**Cantilevers.**

A framework used as a bracket is called a Cantilever.

**17.** Cantilever ABC. AB = 20 ft., CAB = 30°. AC is divided into four equal parts, and the struts are perpendicular to AC.

Load of 5 cwt. at A, 10 cwt. at the three other joints. Find the stresses in the rods and the direction of the reaction at B.

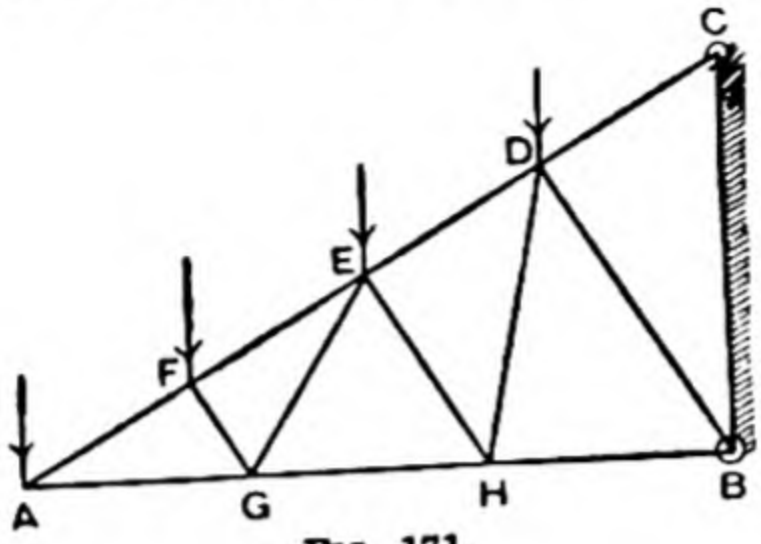



Fig. 171.

*N.B.*—Begin by drawing the triangle of forces for the joint A.

**18.** AB = 15 ft., BD = 10 ft., BC = CD = 7 ft. 10 cwt. at A.

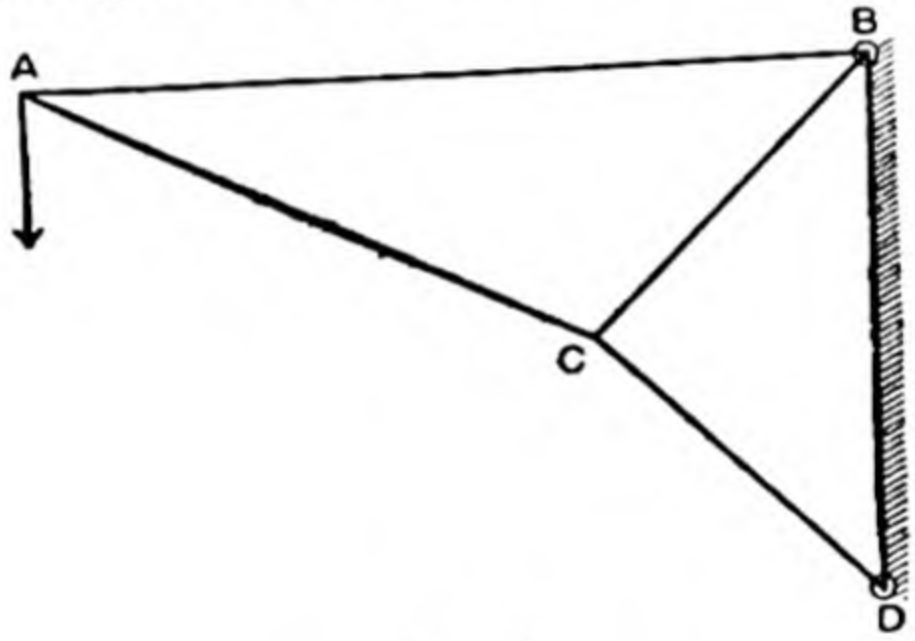



Fig. 172.

**19.** Equilateral triangles. AB = 20 ft. 10 cwt. at A.

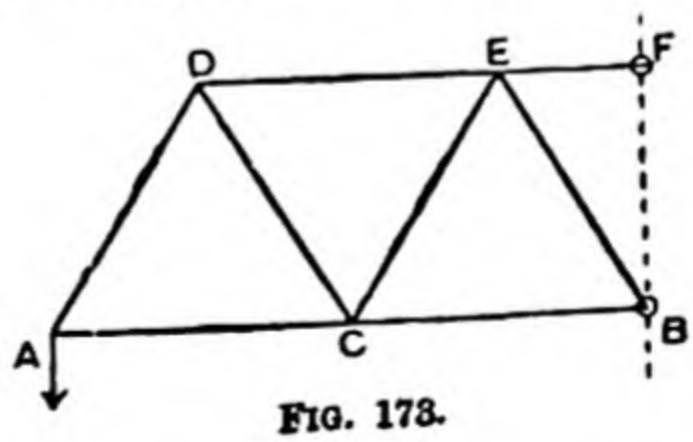



Fig. 173.

**Girders.**

**20.** Warren girder of equilateral triangles. Span AB = 30 ft. 4 tons at D, 5 tons at E, 4 tons at F. From consideration of

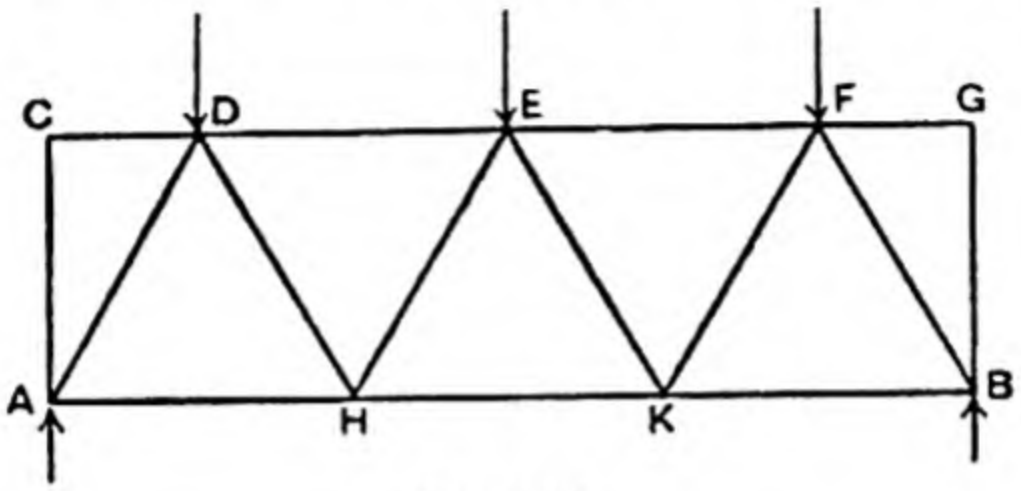



FIG. 174.

the equilibrium of C and G it is clear that AC, CD, FG, BG exert no force, and may be omitted in calculating the thrusts.

**21.** The same girder with 12 tons equally distributed along the four top bars. Show that this is equivalent to 1 ton at C and G, 3 tons at D and F, and 4 tons at E.

**22.** N girder. Span 40 ft. Sixteen tons equally distributed along the top. Slope of braces 45°.

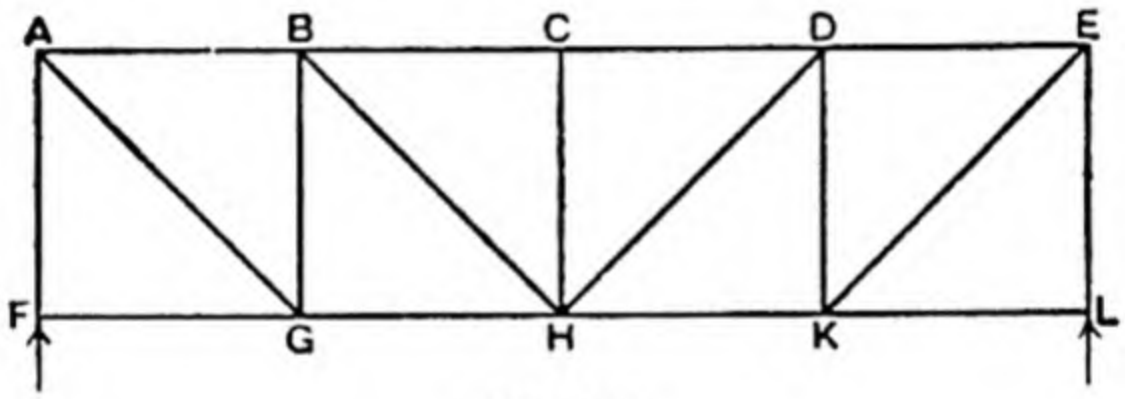



FIG. 175.

**23.** Slope of braces 45°. 10 tons at A, 5 tons at B and at C.

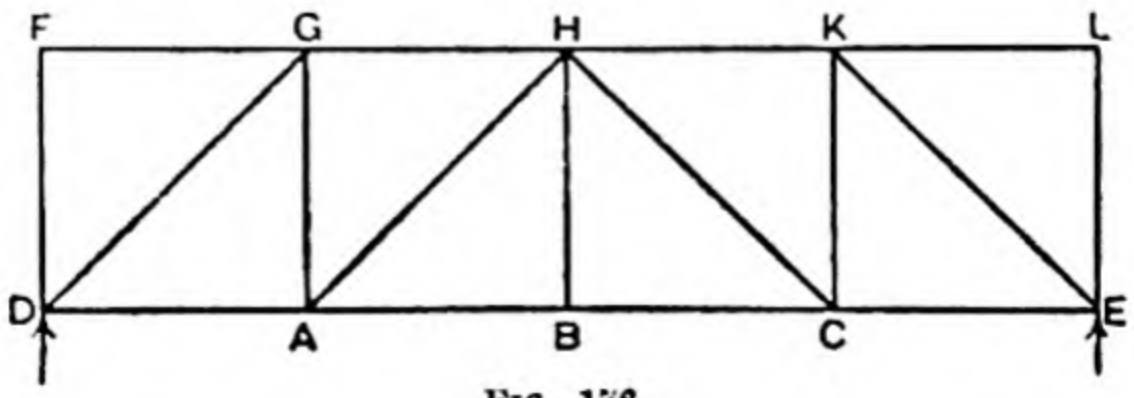



FIG. 176.

**24.** Crane with 10 tons at the mid point of LK. OM makes 30° with the vertical. OM = MK, OP = OK. L is the mid point of MK, and LN is vertical.

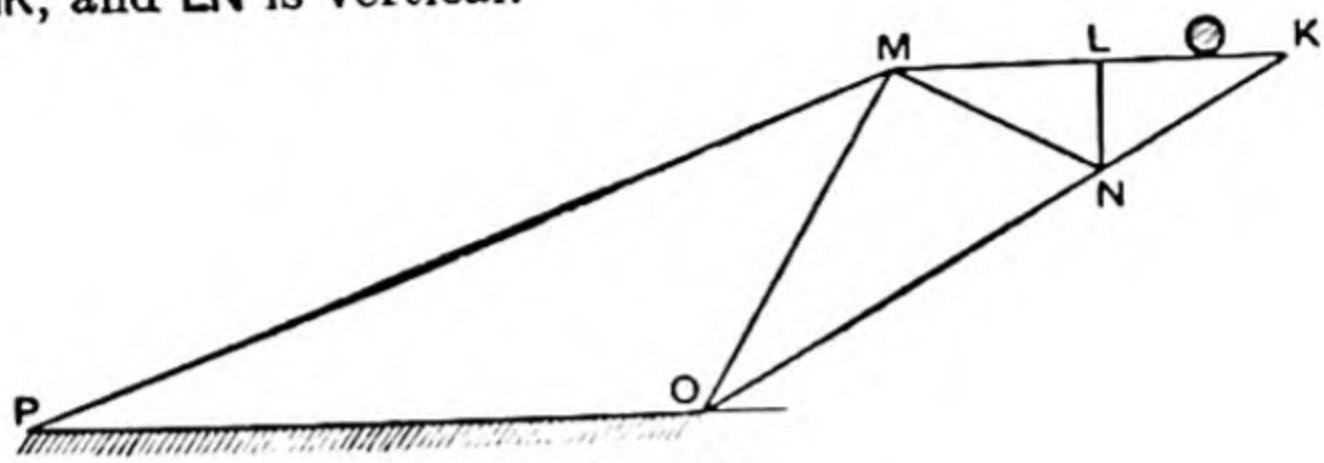



FIG. 177.

**Chains and Suspension Bridges.**

A perfectly flexible chain, suspended from its two ends, assumes the shape of a curve known as the Catenary.

If, however, the dip is small, so that we may assume that the weight of any portion is nearly proportional to its horizontal span, the curve is approximately a parabola.

When the chains of a suspension bridge support a roadway by means of vertical rods, so that the load is distributed equally between the rods, the chain's shape is again a parabola.

The case of a stretched wire in which the load per *horizontal foot of span* is uniform may be investigated as follows:

Consider the equilibrium of the portion AP of the chain, where A is the lowest point.

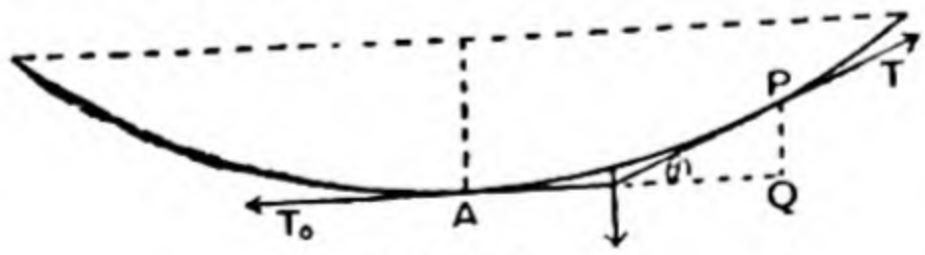



FIG. 178.

Let AQ = $x$, QP = $y$, $w$ = the weight of unit horizontal span.

The forces acting on AP are, $T_0$ the horizontal tension at A, T the tension at P and $wx$ the weight of the chain

AP which acts midway between A and Q, according to the assumption above.

Taking moments about P (Fig. 178), we have

$$T_0 . y = wx . \frac{x}{2};$$

$$\therefore\ y = \frac{w}{2T_0} . x^2,$$

which is the equation of a parabola.

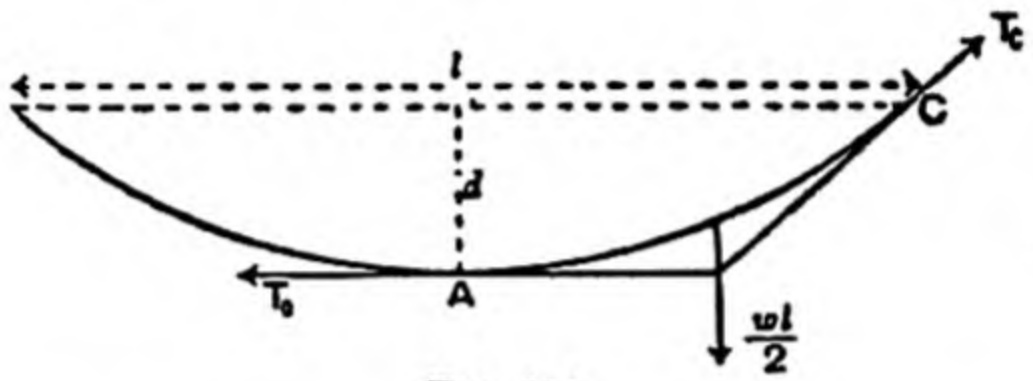



Fig. 179.

Taking moments about C for the forces acting on the chain AC, we have (Fig. 179)

$$T_0 . d = \frac{wl}{2} . \left(\frac{l}{4}\right), \quad i.e.\ T_0 = \frac{wl^2}{8d}.$$

Hence the equation of the curve is $y = \frac{4d}{l^2} x^2$, where $l$ is the span.

Again, draw the triangle of forces for the forces acting on the portion AP; we have

$$\tan\theta = \frac{wx}{T_0}.$$

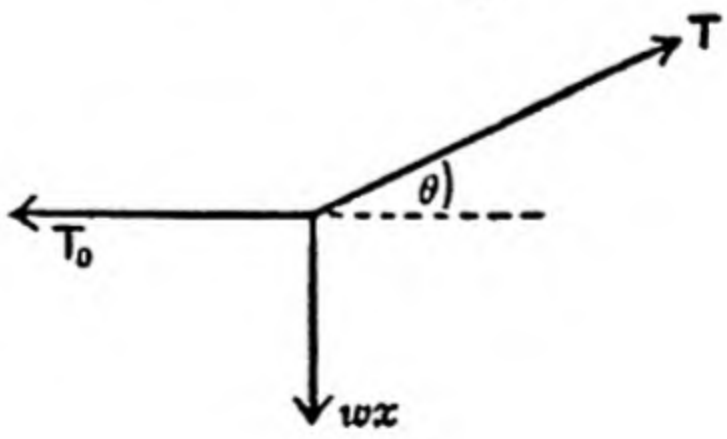



Fig. 180.

Also, since T is the equilibrant of $T_0$ and $wx$,

$$\therefore\ T^2 = T_0^2 + w^2x^2.$$

At C this becomes $T_c^2 = T_0^2 + \frac{w^2l^2}{4}$

$$= \frac{w^2l^4}{64d^2} + \frac{w^2l^2}{4};$$

$$\therefore\ T_c = \frac{wl^2}{8d}\sqrt{1 + \frac{16d^2}{l^2}},$$

which approximately equals $T_0$ if $\frac{d}{l}$ is small.

**Suspension Bridges.**

Suspension bridges, such as those made by military engineers, for spans of 100 ft. and upwards, are designed on the assumption that the load (weight of roadway, etc.) is equally distributed per unit length of span, so that the curve assumed by the cable is a parabola, and details of the stresses may be calculated from the results obtained on p. 186. A suspension bridge is unsuitable for railway traffic, as a large "live" or moving load is likely to cause distortion of the structure.

The following example shows how the question may be treated graphically :

Span 100 ft. Load $\frac{1}{2}$ ton per foot run to be carried by rods attached to the cables at equal intervals along the roadway. The lowest part of the cable to be 10 ft. below the level of the top of the piers. This is called the "dip," it is generally about $\frac{1}{10}$th of the span.

The total load to be carried is 50 tons, so that each cable takes 25 tons.

If there is an odd number of bays, the cable over the middle bay will be horizontal by symmetry.

Consider the equilibrium of the portion of the cable between the mid points of the end bay and the middle bay.

This part of the cable carries a load of $12\frac{1}{2}$ tons distributed among the vertical rods, the resultant acting at the middle rod. (The cables are designed to support the whole weight of the roadway independent of any support at D, and the parts between successive rods are assumed to be straight lines.)

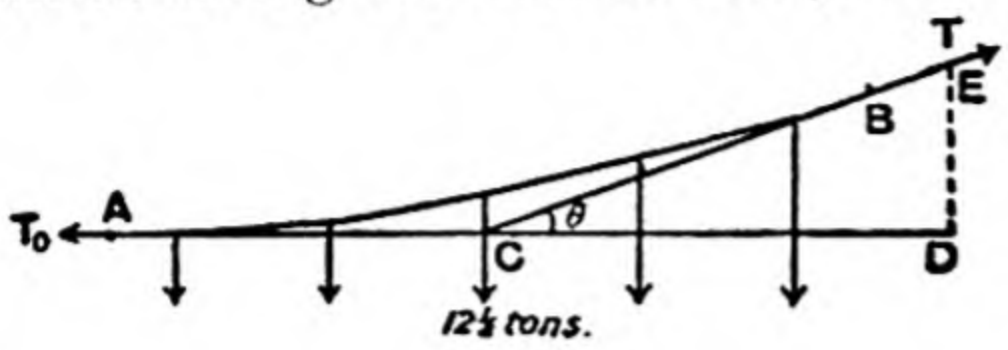


FIG. 181.

The other forces are, $T_0$ the tension in the horizontal section, and T the tension at the highest point.

Since these three forces must meet at a point, the line of action of T must pass through C.

The actual pattern of the bridge may be constructed full size or to scale on the ground.

AD = 50 ft., ED = 10 ft. (the dip). CED is a triangle of forces. ED will represent $12\frac{1}{2}$ tons, and by measuring CD and CE we obtain $T_0$ and T.

We can obtain these values by calculation, for, if there are 11 bays, $CD = \frac{3(100)}{11}$ and $\tan\theta = \frac{ED}{CD} = \cdot 37$ and $T \sin\theta = 12\frac{1}{2}$, hence $T = 36\cdot 3$ tons.

Also $T_0 = T\cos\theta$; $\therefore\ T_0 = 34\cdot 1$ tons.

Using the formula on p. 186, which assumes that C is the mid point of AD, we have

$$T_0 = \frac{wl^2}{8d} = \frac{\frac{1}{2}(100)^2}{8(10)} = 62\cdot 5 \text{ tons},$$

which is carried by two cables, so that $T_0 = 31\cdot 25$ tons.

If we take $\tan\theta = \frac{1}{2}\frac{0}{5}$ in Fig. 181, we also get this value for $T_0$.

In military bridges the roadway is suspended from the cables by " slings " of wire rope. When the curve is a parabola, the length of each sling may be calculated from the equation of the curve $y = \frac{4d}{l^2} x^2$ (see p. 186).

Since the origin is taken at the lowest point of the parabola, the length of a sling above this point at a distance of 20 ft. from the centre will be

$$y = \frac{4(10)}{100^2} \times 20^2 = \frac{8}{5} = 1{\cdot}6 \text{ ft.},$$

so that if the origin is 5 ft. above the roadway, this sling will be of length 6·6 ft. if the roadway is considered horizontal. In practice the roadway is higher at the centre than at the piers; the slope is known as " camber."

A force diagram for the various sections of the cable will enable us to determine the shape of the bridge.

Letter the spaces by Bow's method, and draw the force

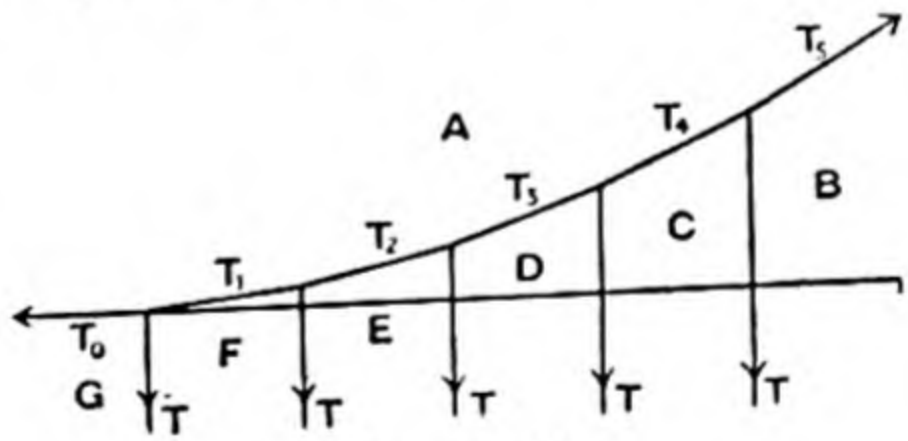



FIG. 182 (*a*).

diagram *bga* for half the cable as before.

$$bg = 12\tfrac{1}{2} \text{ tons}, \quad ga = \mathsf{T}_0, \quad ab = \mathsf{T}_5.$$

Let *bc*, *cd*, *de*, ... represent T, the tension in the vertical rods.

Join *ac*, *ad*, *ae*, ... .

Then *bac* is the triangle of forces for the point (ABC), and so on.

The slopes of the sections of the cable are given by the slopes of *ab*, *ac*, etc.

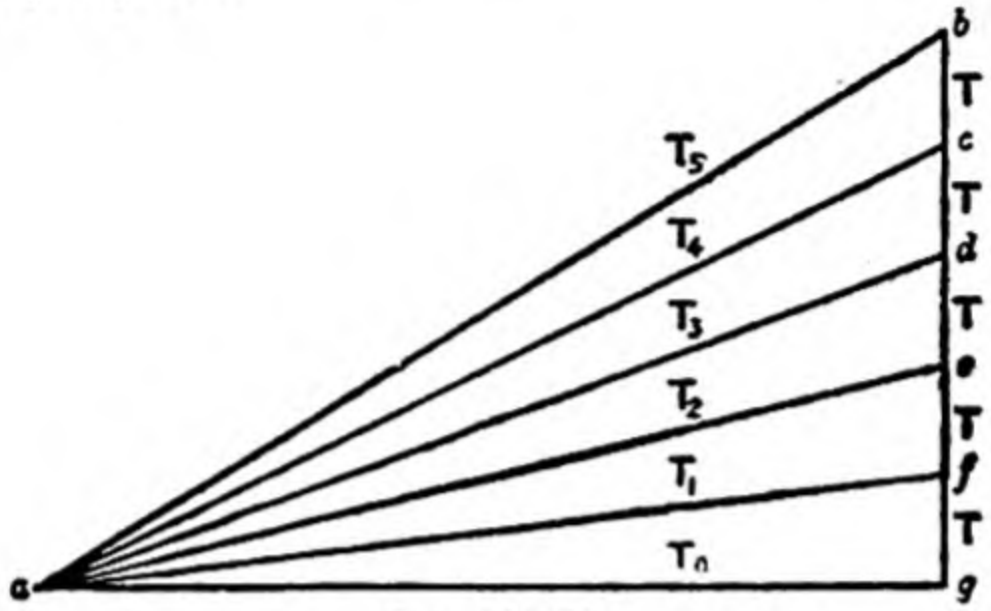



FIG. 182 (*b*).

The cables are not fastened to the tops of the piers, but led over them and anchored in the ground. In the more elaborate bridges the cable passes over a roller at the top of the pier, which allows for a certain amount of play caused by expansion.

## EXAMPLES XXVI.

**1.** A wire weighing $\frac{1}{2}$ lb. per foot length is stretched between two trolley standards 50 ft. apart, by a horizontal tension of $\frac{1}{2}$ ton. Find the sag in the middle of the span.

**2.** Wire weighing ·02 lb. per foot is stretched horizontally between two points 50 ft. apart. What must be the tension in the wire if the sag is not to exceed 2 inches ?

**3.** A suspension bridge of span 200 ft. and dip $\frac{1}{12}$ is to carry a load of 600 lbs. per foot run. Find the tension in the cables at the lowest point and at the piers.

If the lowest point is 5 ft. above the roadway, find the height of the piers and the length of a sling 50 ft. from the mid point.

**4.** A uniform chain, weighing 20 lbs., is suspended from two points in a horizontal line. The tangents to the chain at

these points make 20° with the vertical. Find the tension at the ends by considering the equilibrium (1) of the whole chain, (2) of half the chain.

**5.** If a suspension bridge of 200 ft. span, dip $\frac{1}{10}$, is to be carried by 50 equidistant slings on each cable, each taking a tension of 1 ton, find the tension in the cable at the middle and the pressure on each pier if the cable is led over a saddle and makes the same slope with the pier on either side.

**6.** Draw to a large scale the shape of half the cable of a suspension bridge 110 ft. span, dip $\frac{1}{5}$, to be carried by 10 slings on each cable at equal distances along the roadway, exclusive of the support at the piers, each sling taking the same load. Roadway to be 5 ft. below lowest point of the cable.

**7.** A wire, weighing 1 oz. per ft., is fastened to the top of two equal poles 20 ft. high placed 50 yards apart on the same level. The dip at the mid point is 2 ft. Find the moment of the force tending to break each pole at the ground. If the wire passes over a series of such poles, find the pressure produced on each pole.

**8.** A wire, which must not be stretched by a tension of more than 100 lbs., is carried over a series of poles. If the wire weighs ·03 lb. per foot, and the dip is not to exceed 2 inches, find the greatest distance apart the poles may be placed.

**9.** The span of a suspension bridge is 88 ft., and the roadway is carried by 10 vertical rods on each cable, each rod taking a tension of $\frac{1}{2}$ ton. If the dip is 10 ft., find graphically (1) the tension in each part of the cable between the vertical rods, (2) the inclination of each part to the horizontal.

**10.** A bridge 100 ft. long, dip $\frac{1}{10}$, supported by 9 rods on each cable, each rod carrying a load of $\frac{1}{2}$ ton. Find the tension in each part of the cable. (Here there is no horizontal link; imagine the central rod divided into two portions, each taking $\frac{1}{4}$ ton with a short horizontal link between them.)

# CHAPTER VIII.

## COUPLES. BENDING MOMENTS.

It has been shown in Part I. p. 34, that the moment of a couple about any point in its plane is P$p$, where $p$ is the arm of the couple; see also Ex. IV. (2), p. 36.

**Theorems on Couples acting on a Rigid Body in One Plane.**

I. The resultant of two couples is a couple.

Let the lines of action of the forces (PP), (QQ), forming two couples, meet at A and B.




Fig. 188.

By symmetry, the resultant of P and Q acting at A equals the resultant of P and Q acting at B, also these resultants will be parallel and opposite in direction; hence they form a couple.

II. If two couples have the same moment but are of opposite sign, they balance one another.

Let the lines of action of the forces (PP), forming one couple, meet the lines of action of the forces (QQ) of the other couple at A and B.

Let $a$ be the arm of the couple (PP) and $b$ the arm of the couple (QQ); then $Pa = Qb$, since the couples have the same moment (Fig. 184).

Taking moments about A for the forces P and Q acting at B, we have $Pa - Qb$ = moment of their resultant about A.

Since this is zero, it follows that the resultant passes through A. Similarly, by taking moments about B for the forces acting at A, we see that their resultant passes through B. But the resultant of P and Q at A equals the resultant of P and Q at B, and acts in the opposite direction; since both act along the line AB, they

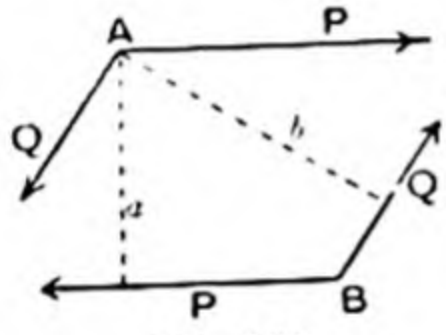



Fig. 184.

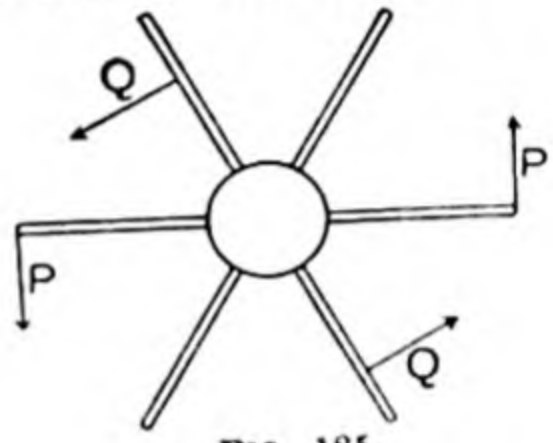



Fig. 185.

neutralise one another. Hence the two given couples are in equilibrium.

It follows that if the direction of the forces (PP) were reversed, the couple (PP) would be equivalent to the couple (QQ). We can therefore substitute for a couple (PP) another couple (QQ) in the same plane, if the couples have the same moment and cause rotation in the same sense.

This may be illustrated from the fact, that if we wish to rotate a capstan, it is immaterial whether we use a couple such as (PP) or the couple (QQ), so long as their moments are equal (Fig. 185).

III. Any number of couples in a plane are equivalent to one couple whose moment equals the algebraic sum of the moments of the given couples.

Let the moments of two given couples be $Pa$ and $Qb$ respectively.

By Chap. II., the couple (QQ) can be replaced by another couple of the same moment and sign, which may be put so that one force (Q) of the couple acts along the same line as one of the forces (P) of the other couple.

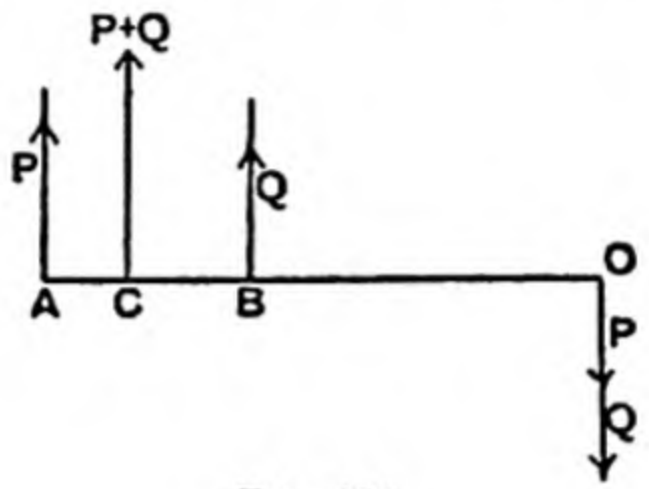



FIG. 186.

The resultant of P at A, and Q at B, is (P + Q) at a point C.

Since the algebraic sum of the moments of P and Q about O equals the moment of their resultant about O, we have

$$P(AO) + Q(BO) = (P + Q)CO,$$

*i.e.* moment of couple P + moment of couple Q
= moment of couple (P + Q).

The theorem can now be extended to any number of couples.

These theorems are also true for couples in *parallel* planes.

To prove that two couples of equal moment in parallel planes will balance if they act in opposite senses.

One of the couples can be replaced by another in the same plane, such that the forces are parallel and equal to the forces of the couple in the other plane.

Join any two points A and B in the line of action of the two forces which act in the same direction.

Draw AC and BD perpendicular to the forces in the planes of the two couples.

Join DC.

Then ACBD is a parallelogram, and the diagonals bisect one another at E.

The forces at C and D are equivalent to 2P at E, and the forces at A and B are equivalent to 2P at E in the opposite direction; therefore they are in equilibrium.

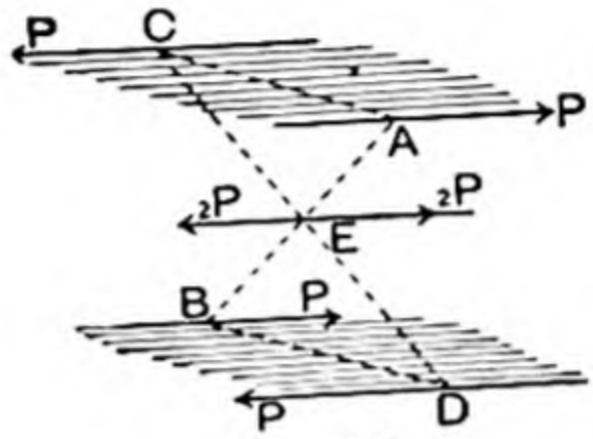



FIG. 187.

It follows therefore that a couple in one plane may be substituted for a couple of equal torque in a parallel plane rotating in the same sense. A system of couples in parallel planes may be compounded as in III., since the given couples may all be transferred to one plane parallel to the given planes without altering their effect on a rigid body.

EXAMPLE 1. Couple acting on a crank. The piston rod AB is constrained to move along the line AO. The connecting rod BC rotates the crank OC. Let P be the thrust produced by the steam in the cylinder on the rod AB.

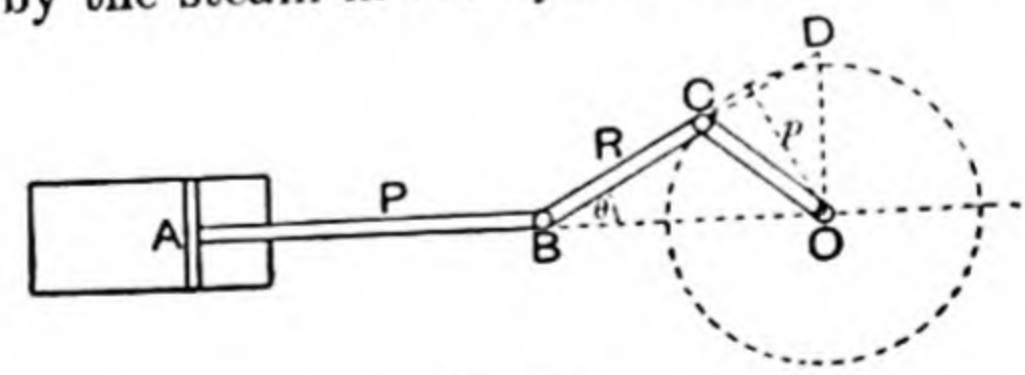



FIG. 188.

If R is the corresponding thrust along BC, we have $R \cos\theta = P$.

If $p$ be the perpendicular from O on BC, the moment of R about $O = Rp = P \sec\theta \,.\, p$. This is equivalent to a couple

of the same moment, for the connecting rod produces a thrust R on the crank at C, and the axle at O will produce an equal thrust R in the opposite direction, otherwise the crank would move bodily. The moment of this couple is $Pp \sec\theta$.

Draw OD perpendicular to BO, then $OD = p \sec\theta$; $\therefore$ the moment of the couple is P . OD, and if P is constant, the torque is proportional to OD.

EXAMPLE 2. A pulley is rotated by a weight at the end of a string which produces a tension T in the string.

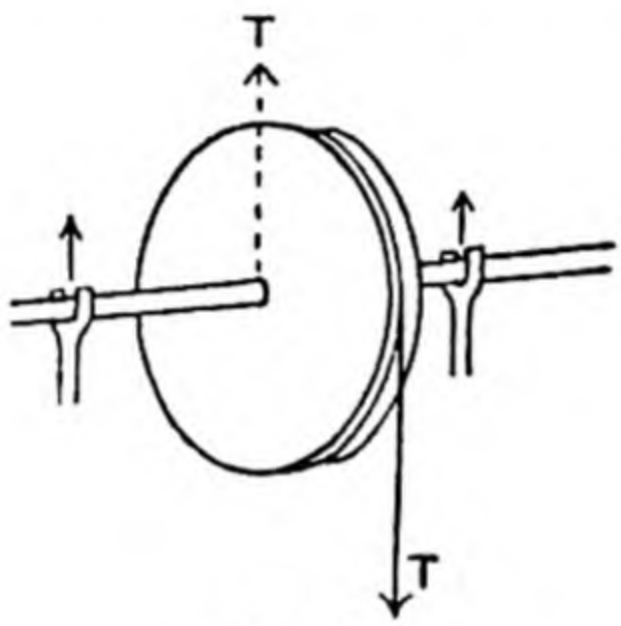



FIG. 189.

The moment of T about the axis of the pulley is $Tr$. Besides supporting the weight of the pulley, the bearings must also exert an upward force equal to T. The torque produced by T is therefore equivalent to a couple of moment $Tr$.

If the pulley does not rotate owing to friction at the axle, or if it rotates with uniform angular velocity, the friction produces a torque whose moment is also $Tr$ in the opposite sense.

EXAMPLE 3. The force P acts at right angles to a horizontal arm AB, which is fixed to a vertical support AC.

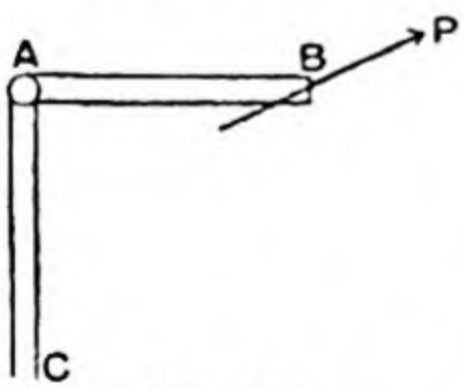



FIG. 190.

The force P will tend to produce a rotation of the arm, and it will also tend to break the vertical rod AC.

If at A we introduce two opposite forces, each equal and parallel to P, the conditions will remain unaltered.

We may now look upon the system as acted upon by a force P at A and a couple whose moment is P(AB).

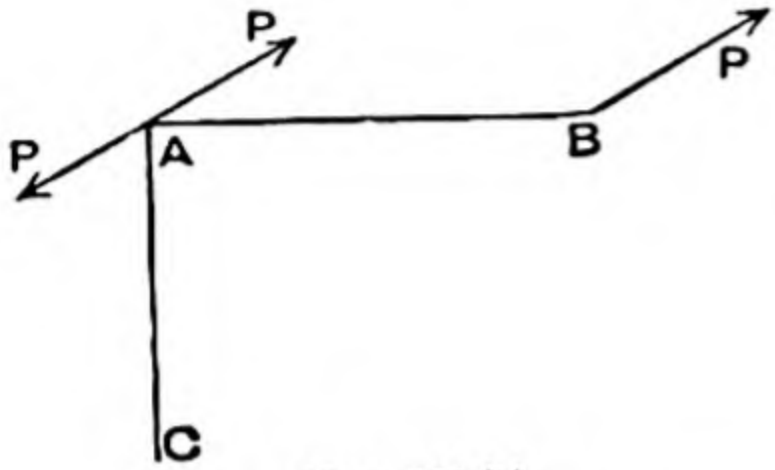



FIG. 191 (*a*).

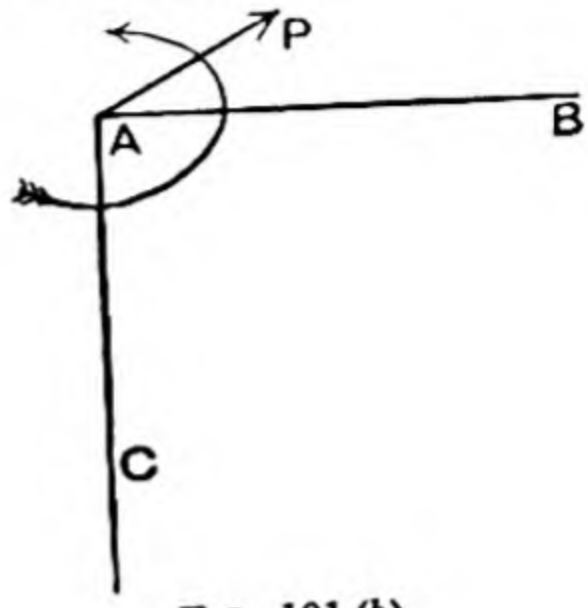



FIG. 191 (*b*).

The force P tends to break AC, and the couple produces the rotation of the arm.

## EXAMPLES XXVII.

**1.** Forces P, Q, P, Q respectively act along the sides BA, BC, DC, DA of the square ABCD, whose side is *a*. Find the moment of the couple which would balance the given forces.

**2.** A light stiff rod AB, 4 ft. long, is fastened to the ground at one end A, so that it is inclined at 20° to the vertical. A weight of 5 lbs. is hung at B. Find the magnitude of the torque which tends to break the rod at A.

**3.** Two equal uniform rods AB, AC, of which AC is 3 ft. long and of weight 2 lbs., are hinged together at A, and held in a vertical plane by a grip at B. The friction at the hinge keeps AC inclined at 40° to AB. If both rods are on the same side of the vertical drawn upwards at A and AB makes an acute angle $\theta$ with it, find the torque of the friction couple at the hinge.

**4.** A rectangular board 12 ft. by 5 ft. is placed in a vertical plane with one corner A on a smooth horizontal plane. If the board weighs 60 lbs., find the couple necessary to support it so that the diagonal through A makes 40° with the vertical.

**5.** One end of a rod is fixed vertically in a vice. A screw is cut at the other end by a tool whose handle is horizontal and 1 ft. long. If the rod projects 8″ from the vice, find the moment of the force tending to break the rod at the vice if

the horizontal force applied at right angles to the handle of the tool is 15 lbs.

**6.** If three forces acting on a rigid body be represented in magnitude, direction *and line of action*, by the sides of a triangle taken in order, prove that they are equivalent to a couple whose moment is represented by twice the area of the triangle.

**7.** Extend the theorem of Question 6 to the case of a system of forces represented in a similar manner by the sides of a polygon.

**8.** A uniform rod weighing 2 lbs., of length 3 ft., can turn about a horizontal axle at right angles to the rod. The radius of the axle is 1″ and its centre is 6″ from the end of the rod. If the rod can remain in a horizontal position when the ends of the axle are fixed, find the friction couple produced by the axle.

**9.** A cord is rolled round the spindle of a heavy flywheel which rotates with uniform velocity when a weight of 20 grams is fastened to the free end of the cord. If the radius of the spindle is 1·5 cms., find the friction couple at the bearings.

**10.** Show that if $\phi$ be the angle OC makes with OB in Fig. 188, p. 195, and if BC$=a$, OC$=r$, the turning moment will be $\frac{Pr\sin(\theta+\phi)}{\cos\theta}$. Calculate the numerical values for $\phi=10°$, $30°$, $60°$, $90°$, $120°$, $150°$, $180°$, when P$=2$, $a=4$, $r=1$; and draw a graph showing the relation between $\phi$ and the couple.

**11.** A pair of compasses with a stiff joint rests symmetrically over a smooth round peg, each leg touching the peg 2″ from the axis of the joint. The weight of each leg being 2 oz. and the C. of G. of each leg 1″ from the joint, find the magnitude of the friction couple at the joint when the legs of the compasses are inclined to one another at 50°.

**12.** To a heavy rod resting on a rough plane two strings are fastened at a distance of 2 ft. apart. The strings are pulled in directions perpendicular to the rod, making the same angle $\theta$ with the vertical on opposite sides of the rod. Find the couple tending to turn the rod if the tension in each string is 5 lbs.

**13.** A bar is suspended horizontally at its extremities by two vertical fibres of equal length, which share its weight between them. If the bar be twisted through an angle $\theta$ (its C. of G. rising vertically), prove that if $\phi$ be the angle the strings now make with the vertical, then $\sin\phi = \frac{a}{l}\sin\frac{\theta}{2}$, where $a$ is the length of the bar, $l$ the length of the fibres. If $w$ be the weight of the bar, prove that the couple required to twist the bar through this angle $\theta$ is $\frac{1}{4}\frac{wa^2}{l}\cdot\frac{\sin\theta}{\sqrt{1-\frac{a^2}{l^2}\sin^2\frac{\theta}{2}}}$.

**14.** A circular disc of weight W and radius $a$ is suspended horizontally by 3 equal vertical strings of length $b$ attached symmetrically to the perimeter. Prove that if T is the tension in each string when the disc is twisted through an angle $\theta$, so that the centre rises vertically, the couple required to keep the disc in this position is $3a\mathrm{T}\sin\phi\cos\frac{\theta}{2}$, where $\phi$ is the angle the string now makes with the vertical. Hence show that the couple equals $\frac{\mathrm{W}a^2\sin\theta}{\sqrt{b^2-4a^2\sin^2\frac{\theta}{2}}}$.

**15.** A horizontal shaft at A in fixed bearings carries an eccentric of radius 1 ft., the distance of whose centre B from

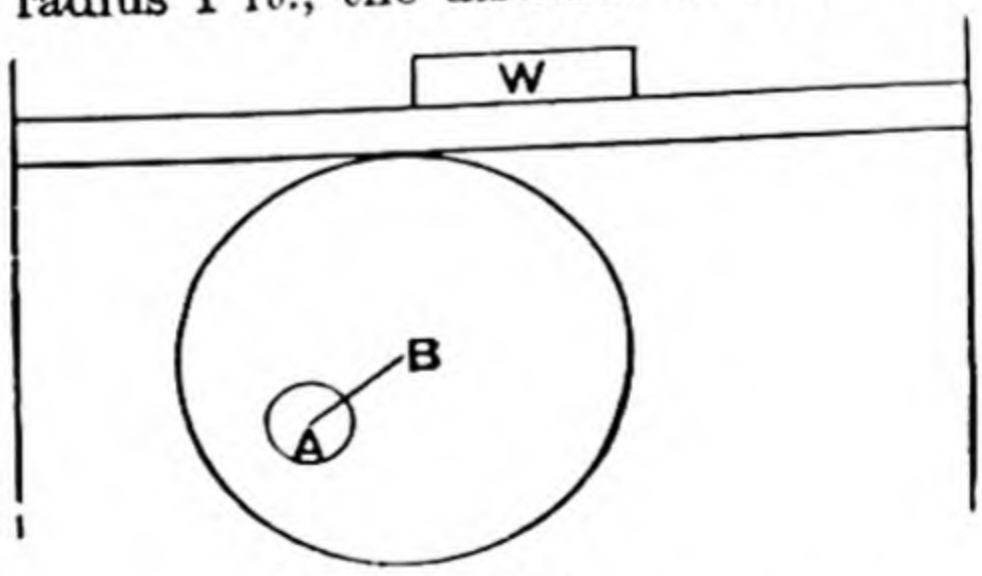



FIG. 192.

A is 6″. The eccentric bears against a bar carrying a load W, which is constrained to move vertically between smooth guides so that it is always horizontal. The shaft is slowly

turned so that W rises. Find the couple that must be applied to the shaft to raise the bar when AB has turned through 120° from the lowest position of B. Weight of bar and load 200 lbs. Coefficient of friction between eccentric and bar = ·25. Neglect the weight of the eccentric.

**Bending Moments and Shearing Force.**

In the introduction, p. 8, it was shown that the forces acting on a body will, in certain circumstances, tend to produce distortion.

If the distortion is inappreciable, the body is assumed to be a rigid body. In practice, however, it is important to estimate the effect of the forces which are tending to break or bend a beam or girder at any given point. This is done by calculating what is known as the Bending Moment and the Shearing Force.

**Bending Moment.**

A beam of weight W is fixed horizontally into a wall. It is clear that the beam is supported by the wall; there

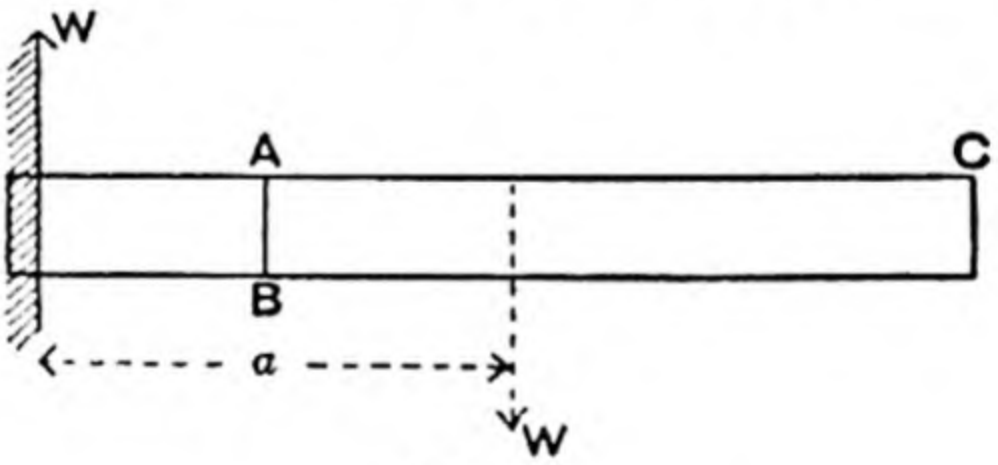



FIG. 193.

must therefore be an upward force equal to W acting on the beam at the wall. These two parallel forces, each equal to W, will form a couple, whose moment is W$a$.

Since the beam is in equilibrium, it must also be acted upon by an opposing couple, whose moment is equal to W$a$.

Consider now the forces acting on any portion AC of the beam.

This part is acted upon by its weight ($w$, say) at the mid point of AC : it is supported by the part of the beam

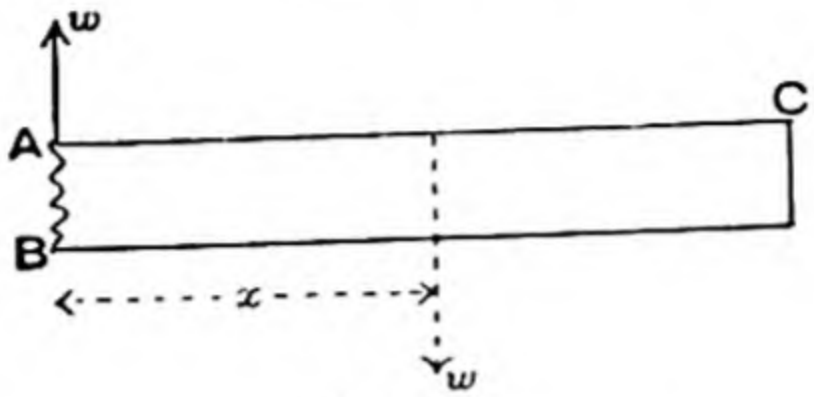



FIG. 194.

to the left of AB, which must therefore be producing an upward force at AB, equal to $w$.

These two equal forces form a couple, whose moment is $wx$. Since the portion AC is in equilibrium, it must be acted upon by another couple in the opposite sense, whose moment is also $wx$. The beam tends to sag, so that the fibres near A are in a state of tension, and those to the

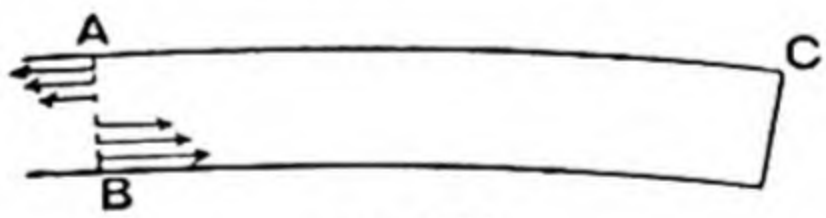



FIG. 195.

left of A produce a force pulling AC to the left. The fibres near B are in a state of compression, and produce a force acting on AC to the right. Since there are no other horizontal forces, the resultant force to the right must equal the resultant force to the left, so that these forces form the couple which is maintaining equilibrium, and its moment must therefore be equal to $wx$.

This moment is called the *Bending Moment* at AB.

**Shearing Force.**

If the part of the beam to the left of AC were unable to exert an upward force equal to $w$, the part AC of the beam

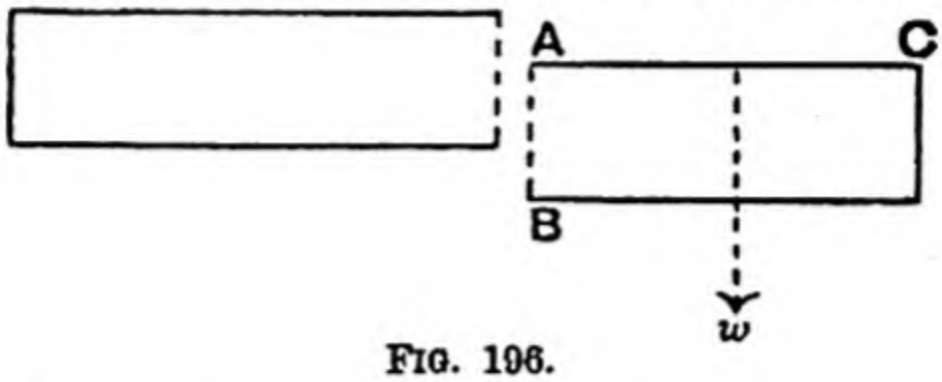



FIG. 196.

would slide downwards. The vertical force $w$ exerted at AB, which prevents the portion AB from sliding across the other portion, is called the Shearing Force at that point.

EXAMPLE 1. A beam AB 12 ft. long is supported at its extremities and loaded with 2 tons at C, 5 tons at D, where AC = 2 ft., DB = 3 ft.

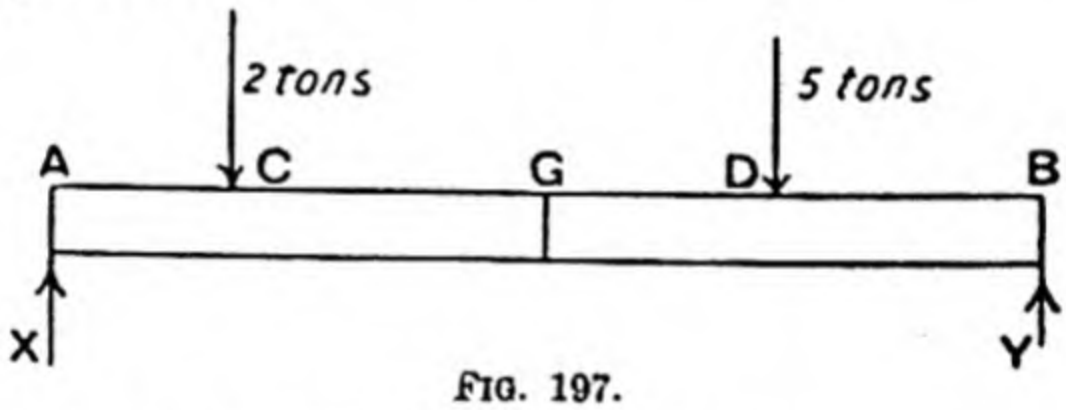



FIG. 197.

To find the B.M. at G, the mid point, assuming the weight of the beam is neglected.

Take moments about B to find X.

$X \cdot 12 = 2(10) + 5(3)$; $\therefore X = \frac{35}{12}$ tons; $\therefore Y = \frac{49}{12}$ tons.

Consider the forces tending to rotate the portion AG about G.

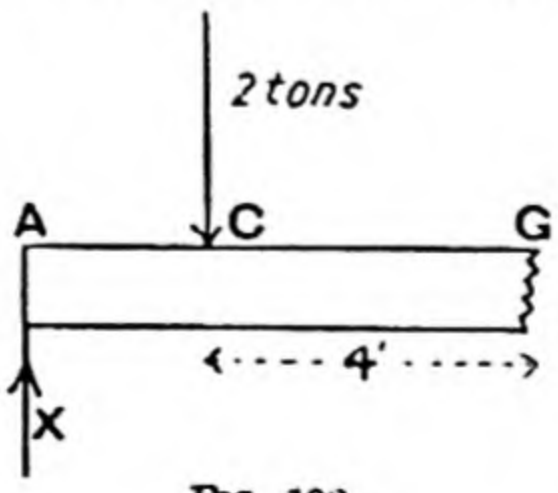



FIG. 198.

The moment of these forces about G will be

$$X(6) - 2(4) = \tfrac{35}{12}(6) - 8 = 9\tfrac{1}{2} \text{ tons-ft.}$$

They are therefore equivalent to a couple whose moment is $9\frac{1}{2}$ tons-ft., which is therefore the B.M. at G.

To find the Shearing Force.

Considering the equilibrium of the part between A and K (K between A and C).

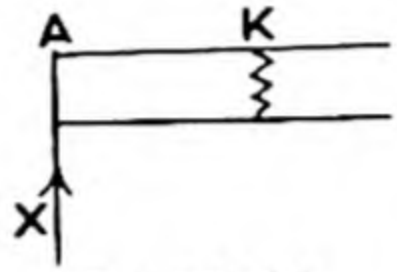



FIG. 199 (*a*).

The force X at A must be neutralised by a downward force equal to X at K; ∴ S.F. from A to C = X = $\frac{35}{12}$ tons.

If K lies between C and D, the force required at K will now

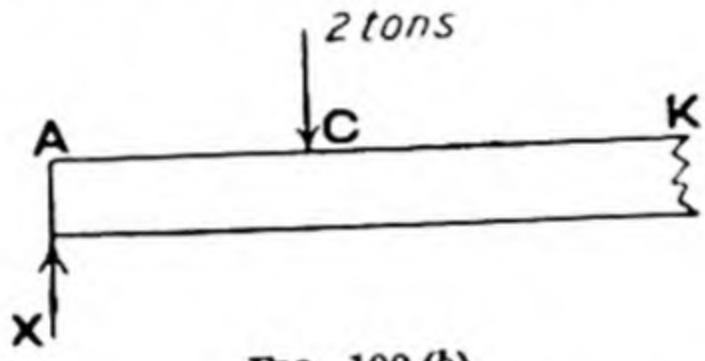



FIG. 199 (*b*).

be $(X - 2) = \frac{11}{12}$ tons; ∴ S.F. from C to D is $\frac{11}{12}$ tons.

If K lies between D and B, the S.F. = Y = $\frac{49}{12}$ tons.

EXAMPLE 2. Suppose the weight of the beam in Qu. 1 to be 1 ton.

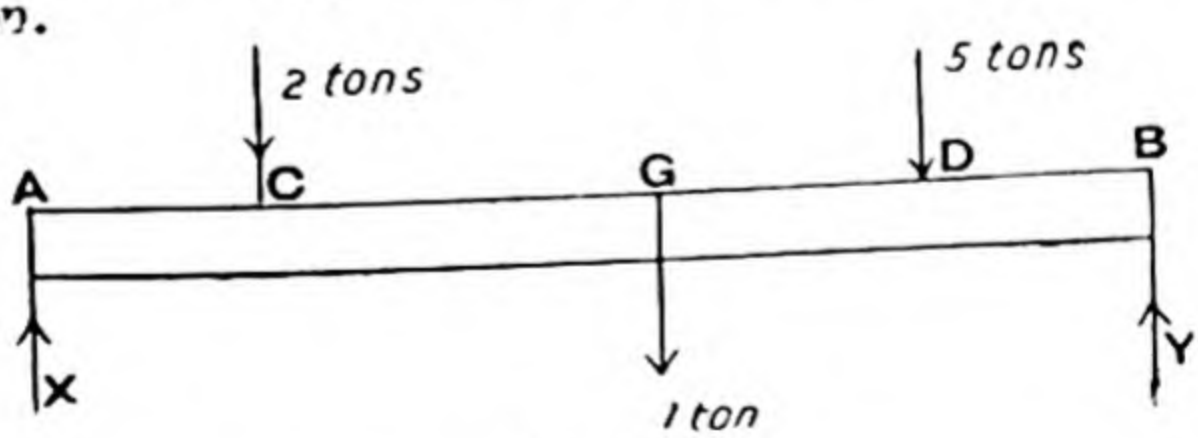



FIG. 200.

Take moments about B to find X.

$$X \cdot 12 - 2(10) - 1(6) - 5(3) = 0. \quad X = \tfrac{41}{12} \text{ tons.}$$

To find the B.M. at G, the mid point.

The forces acting on the portion AG will be X at A, 2 tons at C, and its weight $\frac{1}{2}$ ton at the mid point.

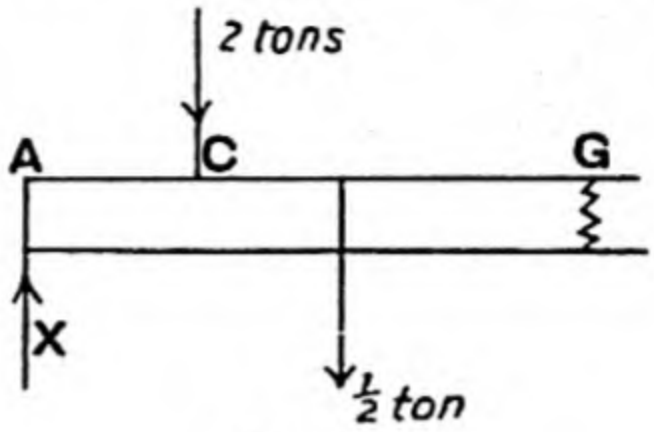



FIG. 201.

The moment of these forces tending to rotate AG about G will be

$$X(6) - 2(4) - \tfrac{1}{2}(3) = 11 \text{ tons-ft.},$$

which is equivalent to a couple of moment 11 tons-ft.

*Shearing Force.*

At any point K between A and C, the portion AK will be acted upon by the reaction X at A, and the weight of $AK = \frac{AK}{12}$ tons at the mid point. The force at K necessary to prevent shearing is therefore $X - \frac{AK}{12} = \frac{41 - AK}{12}$ tons.

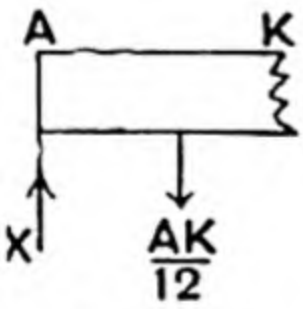



FIG. 202 (*a*).

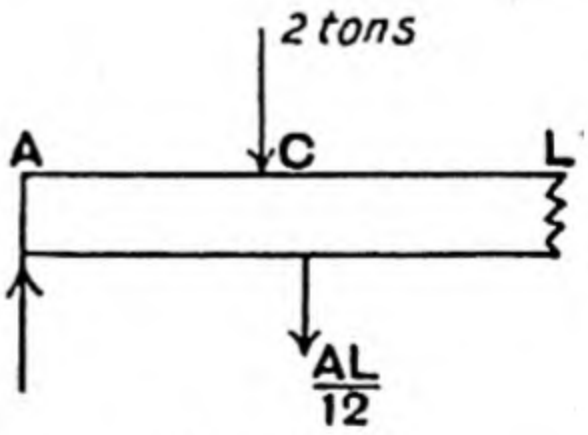



FIG. 202 (*b*).

For a point L between C and G the shearing force will be

$$X - 2 - \frac{AL}{2} = \frac{17 - AL}{12} \text{ tons.}$$

## EXAMPLES XXVIII.

**1.** A uniform beam projects 10 ft. at right angles tc a wall, into which it is fixed. If the beam weighs 3 lbs. per ft. run, find the couple tending to break the beam (1) at the wall, (2) 2 ft. from the wall.

**2.** A beam 10 ft. long supported at both ends carries a load of 1 ton at a point 4 ft. from one end. Neglecting the weight of the beam, find the couple tending to break the beam at its mid point.

**3.** A uniform beam 12 ft. long weighing 60 lbs. is supported at each end. What is the B.M. at a point 8 ft. from one end ? Where is the B.M. a maximum ?

**4.** If a beam of length $l$, supported at both ends, is loaded with a weight W at its mid point, show that the B.M. is greatest at the centre and equals $\frac{Wl}{4}$, but if the beam is loaded with a continuous load of $w$ per unit length, the maximum B.M. is $\frac{wl^2}{8}$.

**5.** A uniform beam 10 ft. long weighs 100 lbs., and is supported at its mid point. What is the B.M. (1) at the ends, (2) at the mid point ?

**6.** A beam 10 ft. long weighs 100 lbs. It is supported at its ends. Find the B.M. and S.F. at the mid section, and also at a section 2·5 ft. from the end.

**7.** A heavy uniform beam AB, whose weight is W, projects horizontally from a wall, to which it is fixed at A. If the length of the beam is $l$ ft., find the B.M. at a point $x$ ft. from the wall.

**8.** A beam of length L ft., supported at its ends, carries a load of $w$ lbs. per ft. length uniformly distributed across the span. Prove that at a point $x$ ft. from the centre the B.M. is $\frac{w}{2}\left(\frac{L^2}{4} - x^2\right)$ and the S.F. is $wx$.

Show also that the B.M. is a maximum at the centre. Where is the B.M. equal to zero ?

**9.** A uniform bar AB, 12 ft. long, weighs 60 lbs., and is loaded with 20 lbs. at its mid point and 20 lbs. at B. It is supported at A and at a point C 4 ft. from B. Find the B.M. at its mid point and the S.F. 3 ft. from A. Where is the B.M. equal to zero ?

**10.** A girder 36 ft. long is supported freely at one end and at a point 6 ft. from the other end. It carries a brick wall which extends along the whole length of the girder and is equivalent to a load of 2 tons per foot run. Find the maximum B.M. and S.F., and determine the points at which they occur.

**11.** A beam 20 ft. long rests on two supports which are 15 ft. apart, the right-hand end of the beam overhanging the support by 5 ft. The beam weighs 44 lbs. per foot, and carries a load of 1 ton at a point $7\frac{1}{2}$ ft. from the left support. Find the load which must be put on the overhanging end so that the B.M. under the load of 1 ton is zero.

**12.** A girder 20 ft. long, supported at its ends, carries a distributed load of 1 ton per foot over 6 ft. of its length, the load commencing 3 ft. from the left-hand support. Calculate the B.M. at intervals of 1 ft., and find from a graph its maximum value.

**13.** A uniform beam AC projects 12 ft. horizontally from a wall into which it is fixed at A. If it weighs 20 lbs. per foot and carries a load of 80 lbs. at B, 4 ft. from A, and 60 lbs. at C, calculate its B.M. at a section 8 ft. from A.

**14.** A light beam ABCD, 24 ft. long, is fastened to supports at B, 4 ft. from A, and at C, 6 ft. from D. A torque of 80 lbs.-ft., acting clockwise, is applied at D and an opposite torque of 100 lbs.-ft. at A. If the reactions at B and C are vertical, find their magnitude, and calculate the B.M. and S.F. at the mid point of the beam.

**15.** If the beam in Qu. 14 weighs 10 lbs. per foot, calculate the B.M. and S.F. at a point 10 ft. from A.

**16.** A pole AB projects vertically 20 ft. from the ground. A wire 30 ft. long, stretched to a tension of 500 lbs., is fastened to A, the top of the pole, and to the ground. Calculate the B.M. at a point 5 ft. from A and also at B.

*N.B.*—The rest of the chapter may be omitted at a first reading.

**Work done by a Couple.**

In Chap. II. we have defined the term work, and explained how to calculate the work done upon a body by a force.

If the forces acting upon a body have a resultant, this resultant causes the body to move in the direction of the force.

If, however, the forces acting on a body are equivalent to a couple, the body is made to rotate.

The work done by a couple may be calculated thus:

Let the couple (PP) rotate the body about a pivot at O.

Draw perpendiculars from O to the line of action of the

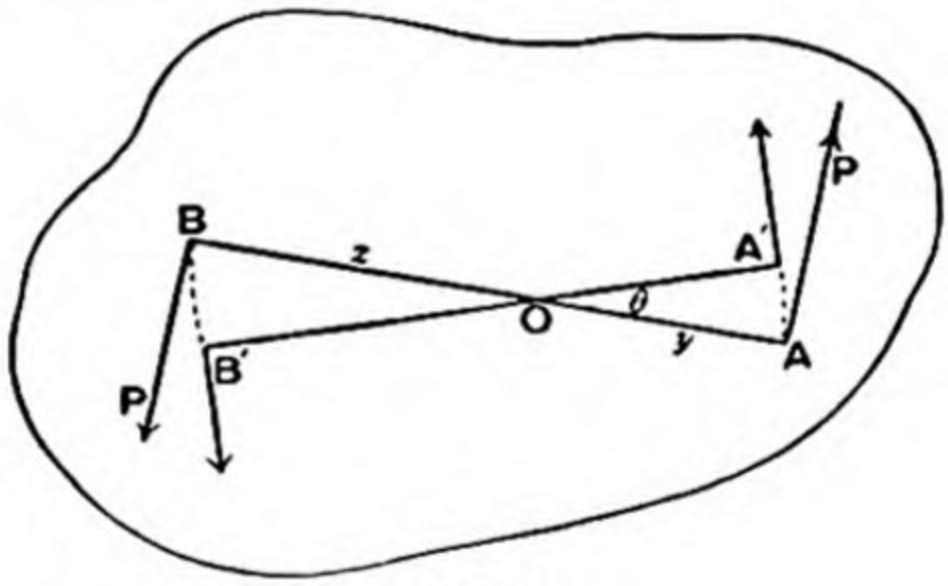



FIG. 203.

forces, and suppose the body to rotate through an angle of $\theta$ radians, the forces remaining constant.

Then work done by the forces comprising the couple

$$= P(AA') + P(BB')$$
$$= Py\theta + Px\theta.$$
$$= P(x+y)\theta$$
$$= C\theta \text{ lbs.-ft.},$$

where C is the moment of the couple, in lbs.-ft., and $\theta$

the radian measure of the angle through which the body is turned.

It has been stated in Part I. p. 41, that a body acted upon by a resultant force will increase its velocity, and that if the body is moving with uniform velocity the resultant force will be zero.

If a body is rotating about an axis under the action of two opposing couples, one of which has a greater moment than the other, the body will rotate more and more rapidly: if the two couples are made of equal moment, the speed of rotation will remain constant.

**Brake Horse-Power of an Engine.**

The engine is made to rotate a wheel, round the rim of which passes a rope which is fastened to a spring balance on one side and supports a load at the other extremity.

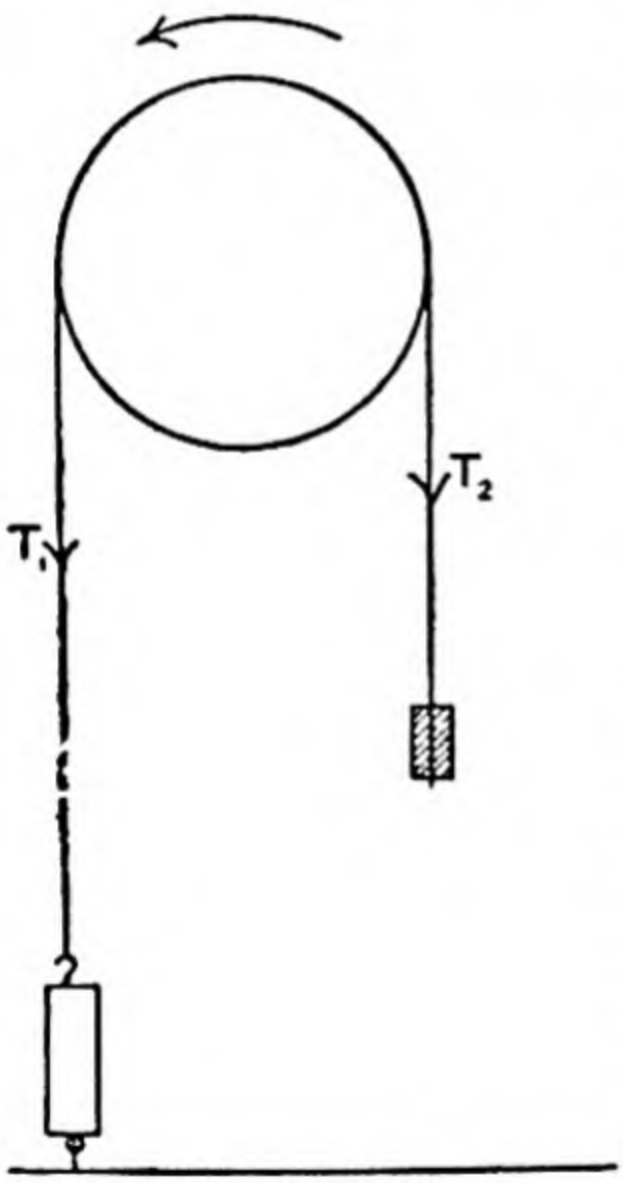


FIG. 204.

The load is adjusted until the engine runs at the speed at which its H.P. is to be calculated.

The engine produces a torque on the wheel, turning it in one sense, and the friction between the belt and the wheel produces a torque in the opposite sense, tending to prevent the rotation of the wheel. The action of the friction is thus equivalent to a couple, and when the wheel runs at uniform speed the torque (C) produced by the engine equals the torque of the friction couple. If the radius of

the wheel is $r$ ft., the moment of the friction about the axle is

$$(F_1 + F_2 + F_3 + \ldots)r = \Sigma F \,.\, (r),$$

and so is equivalent to a couple of this moment.

Let $T_1$ and $T_2$ be the tension of the belt on the two sides of the wheel in lbs. Friction acts clockwise on the wheel, therefore its action on the belt is counter-clockwise.

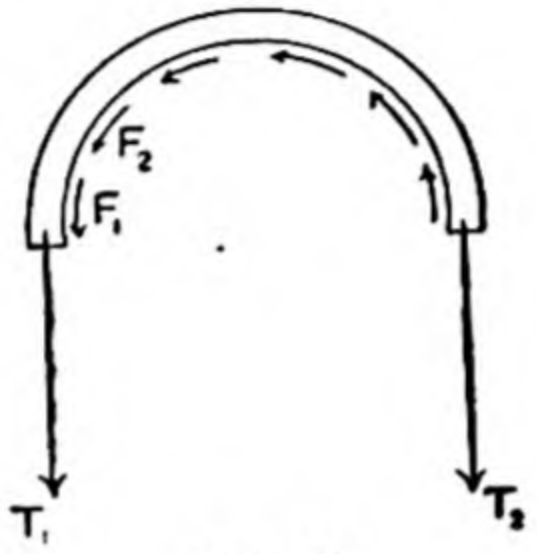



FIG. 205.

If there were no friction, we should have $T_1 = T_2$. But since friction produces a couple whose moment is $(\Sigma F)r$, the moment of $T_2$ about the axis must be greater than the moment of $T_1$ by $(\Sigma F)r$.

$$\therefore\ T_2 r - T_1 r = \Sigma(F)r\ ;\quad \therefore\ (T_2 - T_1)r = C.$$

If the wheel rotates $n$ times per second, the work done by the couple is

$$C \,.\, n \,.\, 2\pi = (T_2 - T_1)r \,.\, 2n\pi \text{ ft.-lbs. per sec. ;}$$

$$\therefore\ \text{H.P.} = \frac{2n\pi r(T_2 - T_1)}{550},$$

since 1 H.P. = 550 ft.-lbs. of work per sec.

**Prony Brake.**

Another method of calculating the brake H.P. of an engine is by means of the Prony brake.

Instead of a belt, a lever is used with a collar at one end, which grips the wheel by friction.

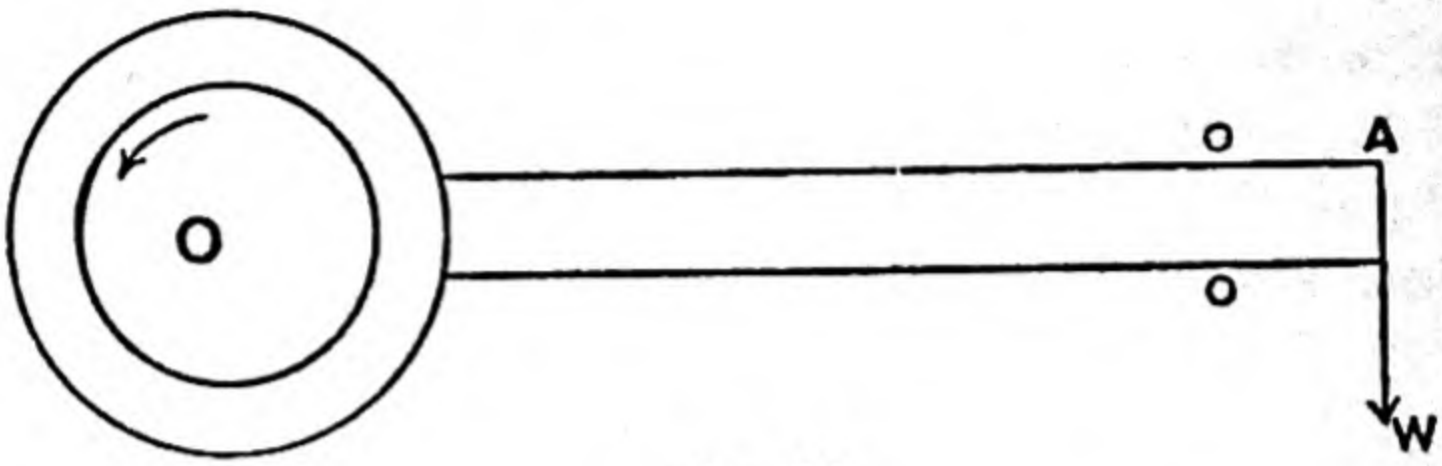



FIG. 206.

A load is added at A until its moment counteracts the torque of the friction which tends to turn the collar round with the wheel. The two stops prevent the lever from flying round before the weight is adjusted.

From considerations similar to those in the previous question, it may be shown that if $AO = a$ and the radius of the wheel is $r$, then $Fr = W \cdot AO$; and that the H.P. for $n$ revolutions per sec. is

$$\frac{2\pi n W a}{550}.$$

EXAMPLE 1. A turbine making 50 revolutions per second develops 500 H.P. Find the torque acting on the driving shaft.

EXAMPLE 2. A motor making 30 revolutions a second exerts a torque of 5·4 lbs.-ft.; find the H.P. it develops.

EXAMPLE 3. A boat is driven by a screw making 100 revolutions per minute. If the resistance is equivalent to 3 tons weight and the speed of the boat is 20 mls./hr., find the torque on the propeller shaft if the efficiency is 50 %.

**Bending Moments (Graphical Method).**

Let the beam, supported at both ends, be loaded as in the diagram.

Using Bow's notation, we draw the funicular polygon and take a pole O, *at unit distance* from *abc*.

Draw the link forces on the original diagram, and find O*d* as before, parallel to $a\delta$ (p. 162).

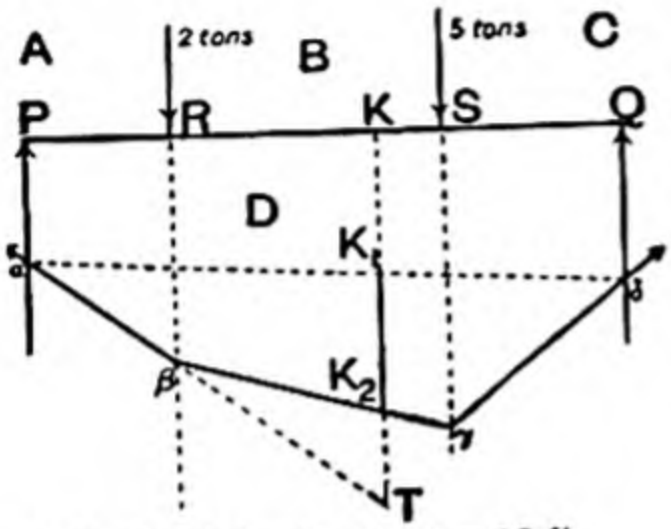



FIG. 207 (*a*).—Scale 1" ≡ 10 ft.

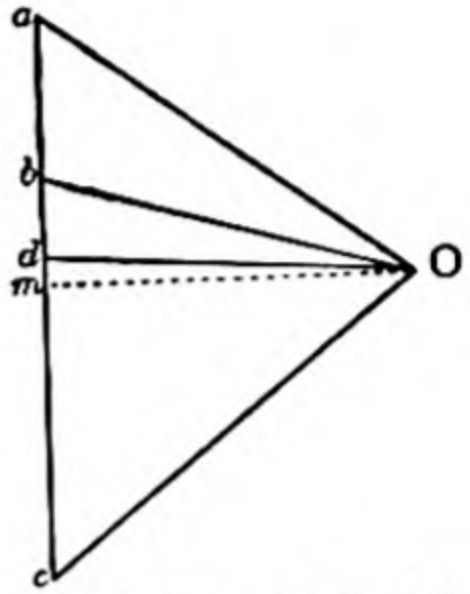



FIG. 207 (*b*).—Scale 1" ≡ 5 tons.

It is required to find the bending moment at K.

Draw a vertical through K to cut the link polygon at $K_1K_2$ and meeting $a\beta$ at T.

The bending moment at K

= moment of force at P – moment of 2 tons

$= da \,.\, PK - ab \,.\, RK.$

Now $$\frac{ad}{TK_1} = \frac{dO}{aK_1} = \frac{Om}{PK}$$

if O*m* is $\perp^r$ to *ac*; $\therefore\ ad \,.\, PK = TK_1 \,.\, Om.$

Also, $$\frac{ab}{TK_2} = \frac{bO}{\beta K_2} = \frac{Om}{RK};$$

$$\therefore\ ab \,.\, RK = TK_2 \,.\, Om;$$

$\therefore$ bending moment $= TK_1 \,.\, Om - TK_2 \,.\, Om$

$= Om \,.\, K_1K_2 = K_1K_2$, since O*m* is unity.

$\therefore$ the bending moment is proportional to $K_1K_2$.

In order to state the value of the bending moment in tons-ft., it is necessary to investigate the scale of the diagram. We have seen that the number of units in the

bending moment is $(Om)(K_1K_2)$, and that this is equal to the number of units in $(da)(PK) - (ab)(RK)$.

If $1'' \equiv 5$ tons on the force scale, $da'' \equiv (da)(5)$ tons.

If $1'' \equiv 10$ ft. for the diagram of the beam, $PK'' \equiv (PK)\,10$ ft.;

$\therefore$ $(da)PK - ab(RK)$ represents a moment of

$$(da \,.\, PK - ab \,.\, RK)\ 50 \text{ tons-ft.};$$

$\therefore$ $(Om)(K_1K_2)$ must be multiplied by 50 to give the B.M. in tons-ft. And if $Om = 1''$, then the B.M. is $(K_1K_2)$ 50 tons-ft.

If the beam tends to become concave downwards, the B.M. is considered to be positive; if concave upwards, the B.M. is negative.

**Shearing Force.**

We can also represent graphically the shearing stress at any point of the beam.

A base line PQ, equal in length to the line which represents the length of the beam, is drawn so that this line when produced passes through the point $d$ of the force diagram.

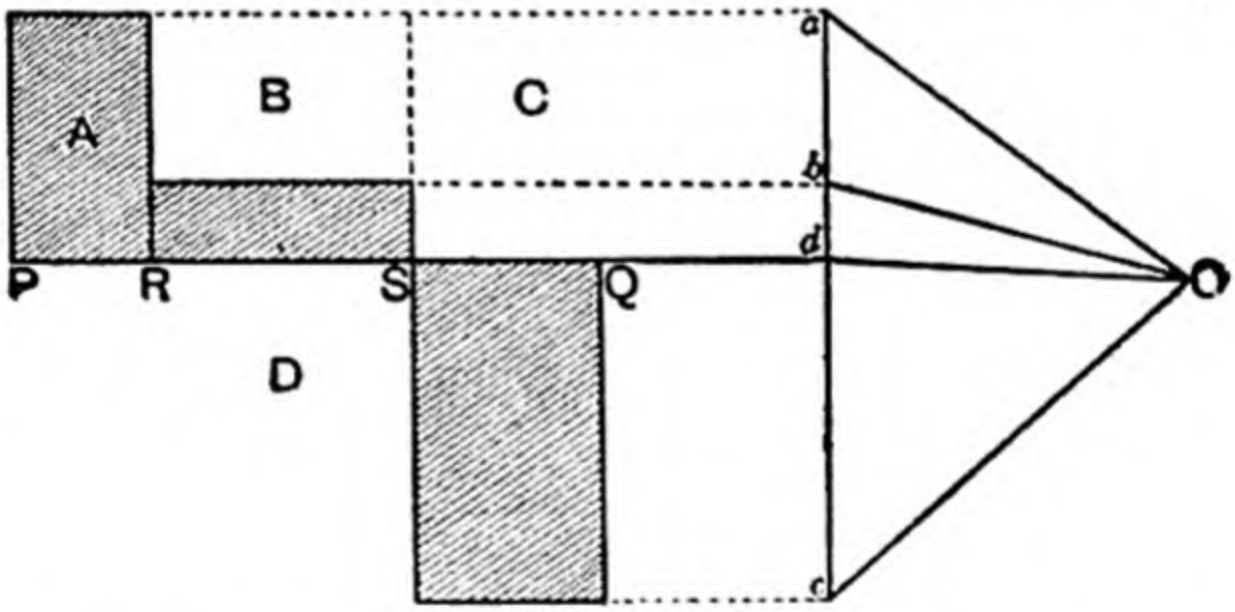



FIG. 208.

The shearing stress from P to R, the point of action of the force (AB), is proportional to $ad$, the force acting at P. From R to S the shearing force is proportional to $da - ab$, *i.e.* $db$, and from S to Q the shearing force is represented by $dc$.

Through $a$, $b$, $c$ draw parallels to PQ so as to form the areas, as shaded in the diagram.

The shearing force along the beam is then represented by the altitude of these rectangles. A convention is made according to which the shearing force is positive and the diagram drawn above the base line so long as the section of the beam tends to move *upwards* relatively to an adjacent section to the right.

It will be noticed that no statement can be made as to the value of the shearing force at a point where there is a concentrated load.

EXAMPLE 1. A beam $l$ ft. long is loaded uniformly with $w$ tons per foot run, and supported at its extremities. To draw its B.M. diagram. Let C be a point $x$ ft. from the centre.

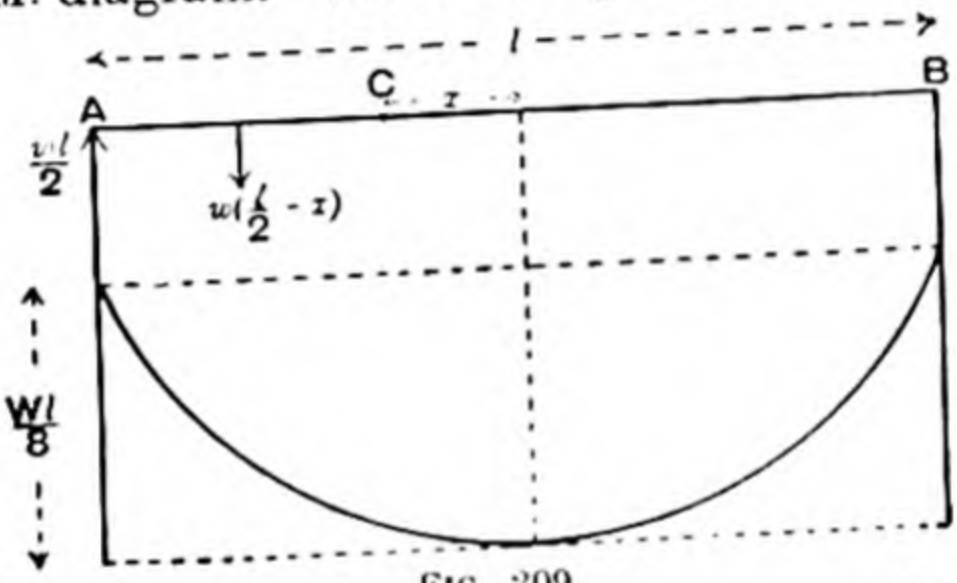



FIG. 209.

The forces acting on the portion AC will be $\frac{wl}{2}$ at A, and the weight of AC $= w\left(\frac{l}{2} - x\right)$.

The B.M. at C will be

$$\frac{wl}{2}\left(\frac{l}{2} - x\right) - w\left(\frac{l}{2} - x\right)\frac{1}{2}\left(\frac{l}{2} - x\right) = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$

At the mid point this equals $\frac{wl^2}{8}$ or $\frac{Wl}{8}$ if $W = wl$.

The curve represented by $y = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right)$ is a parabola, and the value of the B.M. can be plotted by substituting values of $x$ from $x = 0$ to $x = l$.

If the beam also carries concentrated loads, the B.M. for these is found separately as in Ex. 2, and the B.M. for the

two combined is found by adding the B.M. on any point due to the distributed load to that due to the concentrated load.

*Otherwise.* Instead of the method given above, a solution which is purely graphical may be obtained by supposing the beam divided into a number of equal parts, the weight of each part acting at the mid point. A force diagram is then constructed for these concentrated loads as in Ex. 2.

EXAMPLE 2. A beam 20 ft. long loaded as in the diagram with 3 tons, 2 tons, 1 ton, placed 4 ft., 12 ft. and 16 ft. respectively from the end.

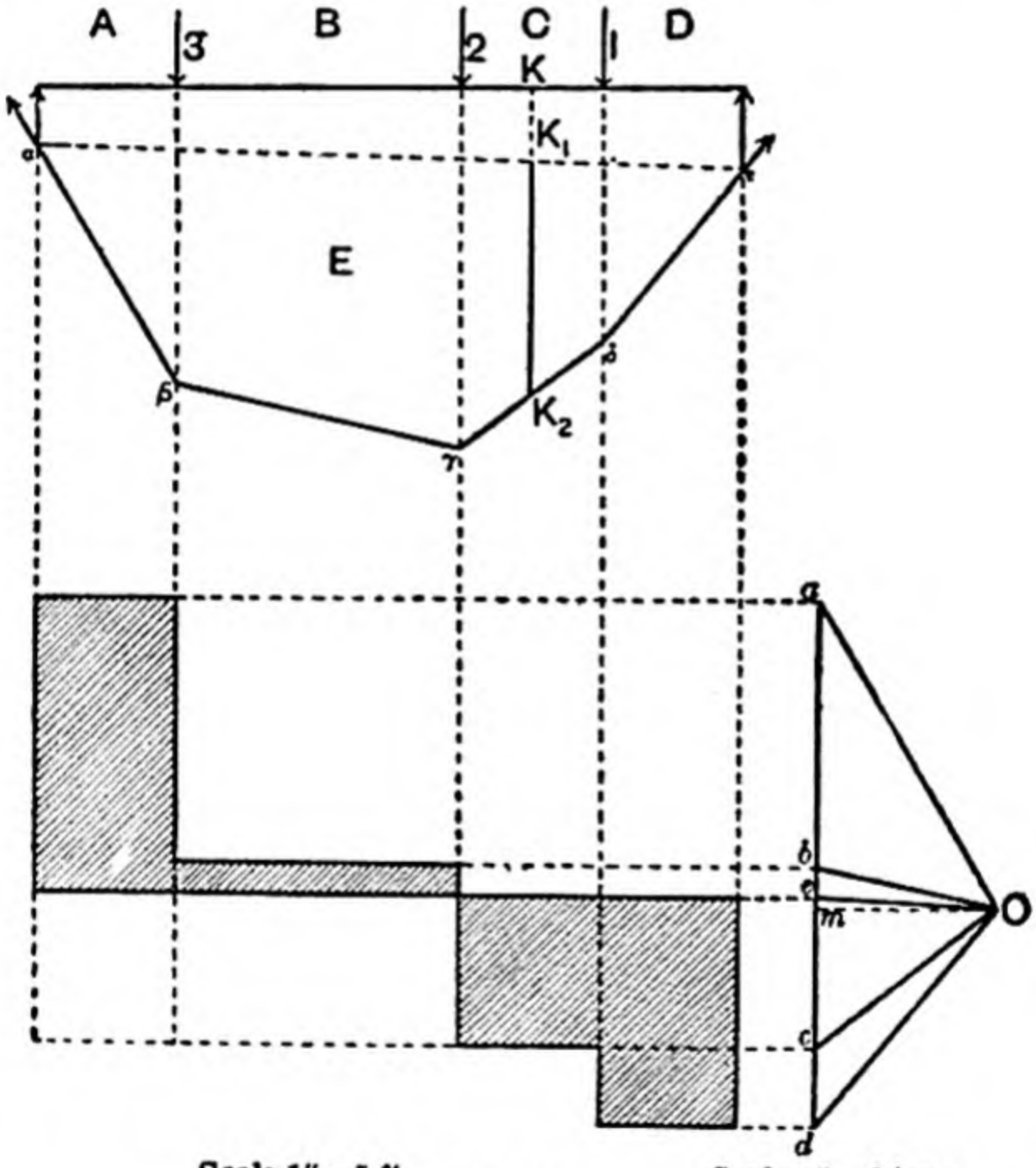



FIG. 210.

Draw a line to represent the beam. Scale $1'' \equiv 5$ ft.

Letter the spaces as in Bow's method.

Draw the force diagram far enough below to leave room for the link polygon. Scale $1'' \equiv 2$ tons.

Take a pole O, 2″ from *ad*.

Through $\beta$ on (AB) draw $\beta\alpha$ parallel to O$a$ and $\beta\gamma$ parallel to O$b$. Complete the link polygon as on p. 162. Draw O$e$ parallel to $a\epsilon$. Then *de*, *ea* represent the reactions (ED) and (EA).

The B.M. at K is $K_1K_2(Om)$

$$= 10(K_1K_2)(Om) \text{ tons-ft.}$$

$$= 20\,(K_1K_2)(\text{tons-ft.}), \text{ since } Om = 2 \text{ units.}$$

Draw a B.M. diagram in which the ordinates represent the values of $20K_1K_2$.

S.F. Draw a base line parallel to the beam to meet *ad* at *e*. Through *a*, *b*, *c*, *d* draw parallels.

Shearing force from left end to (AB) is the force *ea*,

from (AB) to (BC) $= ea - ab = eb$,

from (BC) to (CD) $= ea - ab - bc = ec$,

from (CD) to end $= ed$.

## EXAMPLES XXIX.

**1.** A beam 12 ft. long is supported at its ends, and is loaded with 3 tons at a point 2 ft. from the left-hand support. Draw the B.M. and S.F. diagrams. What is the B.M. and S.F. at the mid point of the beam ?

**2.** A beam AB of 15-foot span supported at its ends is loaded with 3 cwt. 5 ft. from A and 5 cwt. at 8 ft. from A. Draw the B.M. and S.F. diagrams, and find the B.M. at the middle.

**3.** A uniform beam 10 ft. long weighs 400 lbs., and is supported at its extremities. Find graphically its bending moment at intervals of 1 ft.

Assume the bar to be divided into length 1 ft. each, the weight of each portion acting at the mid point of each.

**4.** A light beam 10 ft. long is loaded with 400 lbs. at its mid point, and is supported at its extremities. Draw its B.M. and S.F. diagrams.

**5.** A beam 20 ft. long, supported at its ends, has a load of 10 tons uniformly distributed over it. Construct the B.M. diagram.

**6.** A beam 30 ft. long, supported at its ends, carries a load of $\frac{1}{2}$ ton per foot uniformly distributed, and also a concentrated load of 5 tons, 10 ft. from the left support. Draw its B.M. diagram.

**7.** A horizontal beam AB, 8 ft. long, with one end A fastened to a vertical wall, has 5 tons concentrated at its mid point and 4 tons at B. Construct its B.M. and S.F. diagrams.

**8.** A uniform bar AB, 12 ft. long, weighs 40 lbs. and carries a load of 20 lbs. at its mid point and 60 lbs. at B. It is supported at A, and at C a point 2 ft. from B. Draw its B.M. diagram, and find from it where the B.M. is zero. (See Ex. XXIII. 5.)

**9.** A beam projects 16 ft. horizontally from a wall, and helps to carry a balcony by means of battens placed across it at intervals of 4 ft. The loads are 300, 600, 600 and 400 lbs., measured from the wall. Draw the B.M. and S.F. diagrams, and give their values at a section 10 ft. from the wall.

**10.** A beam 20 ft. long, supported by vertical forces at its extremities, has a distributed load of 4 cwt. per foot run extending from 4 ft. to 12 ft. from one end. Construct its B.M. diagram.

**11.** A beam ABCDE, 30 ft. long, is supported at B and at D (AB = 5 ft., AD = 20 ft.). It carries a load of 4 tons at A, 3 tons at E, and 2 tons at C, a point 10 ft. from A. Draw the B.M. and S.F. diagrams, and find the B.M. and S.F. at the mid section.

**12.** ABC is a straight horizontal beam resting upon two supports A and B, 80 feet apart, the overhanging portion BC being 40 ft. long. The girder carries a load of 1 ton per foot run distributed over its whole length, a concentrated load of 10 tons at C and a concentrated load of 20 tons at the mid point of AB. Construct the bending moment for the beam to the following scales :

$1'' \equiv 400$ ft.-tons, $\quad 1'' \equiv 10$ ft. (C.U.)

## MISCELLANEOUS EXAMPLES.

### CHAPTERS I.-VIII.

### O.

**1.** A horizontal beam AB, 20 ft. long, rests on two supports, one at A and the other at C, 15 ft. from A. The beam carries loads of 2, 4, 6 and 8 tons at points respectively distant 4, 8, 12 and 16 ft. from A. Find the thrust on each support.

Find also the vertical downward force which must be applied at B in order to relieve the support A of its load and the vertical upward force at B to relieve the support C of its load.

**2.** Two weights, each of 2 lbs., are attached to different points of a string which hangs from a support. The upper weight is acted on by a horizontal force of 4 lbs. to the left, and the lower weight by a horizontal force of 1 lb. to the right. Find graphically the tension of each portion of the string when in equilibrium and the angle which each portion makes with the vertical.

**3.** Find the position of the C. of G. of the portion required to complete the square as shown by the dotted lines.

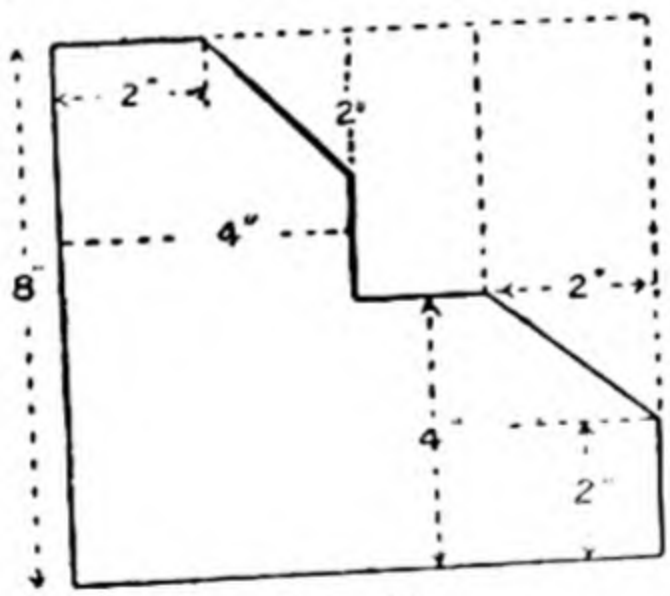



FIG. 211.

**4.** A plank is placed across a horizontal cylinder of radius 2 ft., at right angles to its axis, and then made to oscillate. If the angle of friction between the plank and cylinder is 40°, find the maximum length of plank which may come in contact with the cylinder if it swings without slipping.

**5.** AB and CD are two levers freely hinged to the fixed points B and D. The upper end of CD and the point E on AB

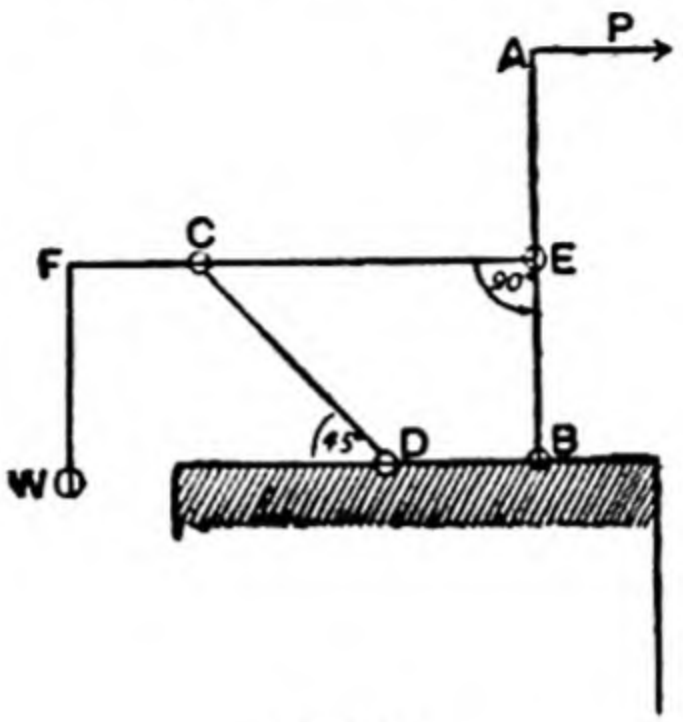



FIG. 212.

are hinged to the link EF. A vertical force W, applied at F, is balanced by a horizontal force applied at A. Prove that

$$\frac{P}{W} = \frac{EF}{EC} \cdot \frac{BE}{BA}.$$ (C.U.)

## P.

**1.** Two strings AB, AC pass over smooth pulleys in the same vertical plane, and carry weights of 2 lbs. and 3 lbs. respectively. A third weight W is suspended from A, and a horizontal

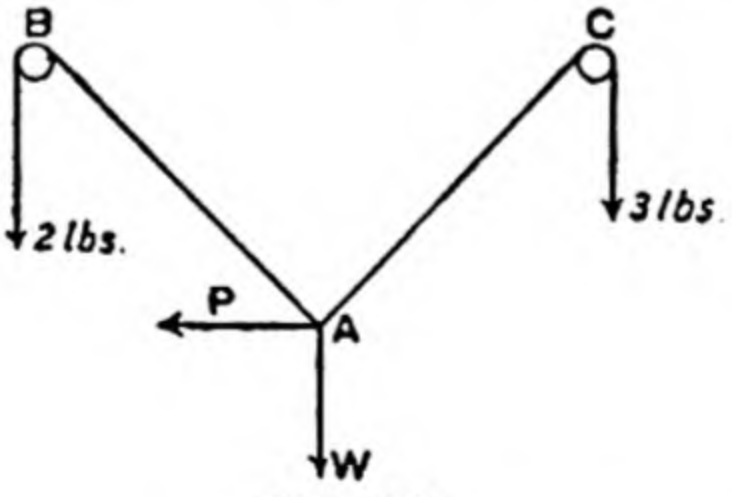



FIG. 213.

force P is also applied at this point. Find graphically the values of P and W when the strings AB, AC each make an angle of 45° with the vertical.

**2.** The beam of a balance is of mass $m$ and length $2l$; when the beam is in equilibrium, the knife edges from which the pans are suspended and the C. of G. of the beam are respectively $d$ and $h$ below the central knife edge. If each scale pan with its load has a mass M, prove that an additional mass $\mu$, placed on one pan, causes a deflection $\theta$ where

$$\tan\theta = \frac{\mu l}{mh + (\mu + 2M)d} \qquad \text{(C.U.)}$$

**3.** Two fixed bars in a vertical plane are each inclined at 45° to the vertical. The ends of a string are tied to rings, each of weight $W_1$, which slide without friction on the bars. From the string a weight $W_2$ is hung. Prove that each half of the string will make an angle $\theta$ with the vertical given by

$$\tan\theta = 1 + \frac{2W_1}{W_2}.$$

**4.** A weight rests on a horizontal table and is pulled by a string which is inclined at 45° to the vertical. If $\mu = \frac{1}{4}$, prove that the weight will just slip when the tension in the string slightly exceeds 0·28 times the weight.

**5.** Find the horizontal force X and the vertical force Y, and

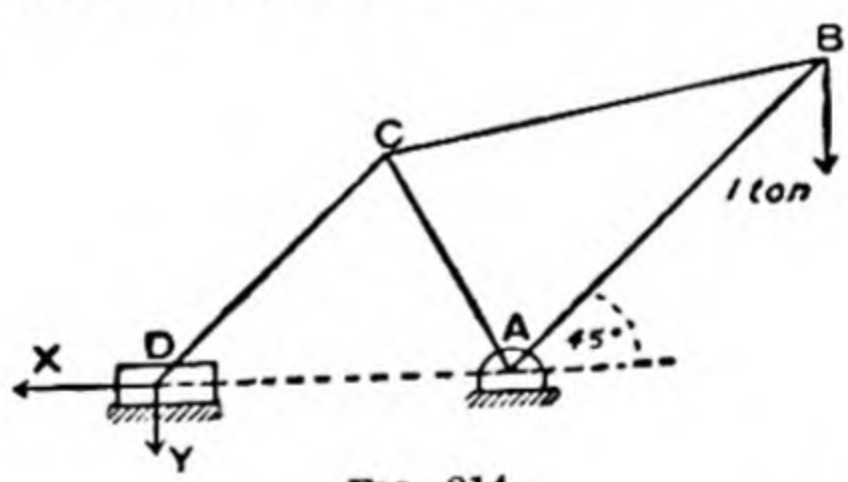



FIG. 214.

the stresses in the frame when A is fixed and 1 ton supported at B. AB = BC = 20 ft.; AC = 9 ft.; CD = 10 ft. (C.U.)

## Q.

**1.** A heavy uniform rod is supported in a horizontal position by three equal forces in a vertical plane, one acting at one end and the other two at the other end. Show that the angle between the directions of the two latter must be 120°.

**2.** Two equal heavy spheres of 1 in. radius are in equilibrium within a smooth spherical cup of 3 in. radius. Show that the pressure between the cup and the sphere is double the pressure between the two spheres.

**3.** Two heavy uniform beams AB, BC, each of weight 10 lbs., are smoothly jointed at B. There is a hinge at A, and the end

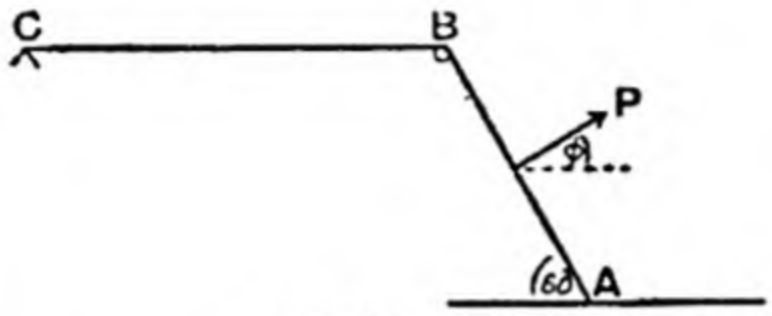



FIG. 215.

C rests on a fixed support. If equilibrium is maintained by a force of P lbs. weight at the mid point of AB, at an angle $\phi$ with the horizontal, prove $P = 10 \sec(30 - \phi)$. (C.U.)

**4.** A horizontal bar, 10 ft. long, carries weights 4 lbs., 5 lbs., 6 lbs., 7 lbs. at distances respectively 2 ft., 5 ft., 7 ft. and 9 ft. from one end; give graphical constructions (1) for the resultant of the weights, (2) for the pressures on the supports, when the bar is supported at its two ends, neglecting the weight of the bar itself.

**5.** A ladder on a horizontal floor leans against a vertical wall. Show that if its C. of G. is at its middle point, the greatest inclination to the wall which is consistent with equilibrium is $\tan^{-1} \frac{2\mu}{1-\mu^2}$, where $\mu$ is the coefficient of friction for the contact both with the floor and with the wall, and that in this position the lines of action of the resultant forces acting at the foot and at the top of the ladder are at right angles to one another.

## R.

**1.** If ABCD is a rectangle, such that the diagonal AC is twice the side AB, and forces of 2 lbs., 4 lbs. and 8 lbs. act along AB, AC and BD respectively, prove that their resultant is parallel to BC, and find its position.

**2.** Two rods, AB, BC, are smoothly jointed at B; the end A is fixed, and AB can turn round it in a vertical plane. C can

move in a horizontal line passing through A. When a force P acts at C along CA, prove that the system can be kept in equilibrium by a couple P$d$ acting on AB, where $d$ is the length of the line drawn from A perpendicular to AC to meet BC.

**3.** A smooth wedge ABC is free to move on the horizontal plane BC. ABC = 30°. A weight of 10 lbs. lies at D on the face AB, and is fastened to a point E by the string DE inclined

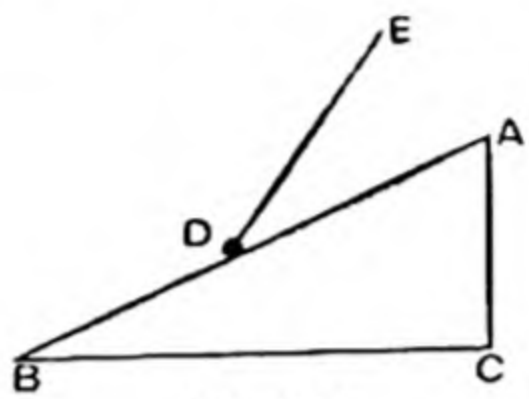



FIG. 216.

at 40° to the vertical, DE being in the same plane as the section ABC. Find the magnitude of the horizontal force which must be applied to AC to prevent the wedge moving.

**4.** The figure shows a chain sling for lifting blocks of stone. The chain ABC passes over pulleys in the hooks at B and C.

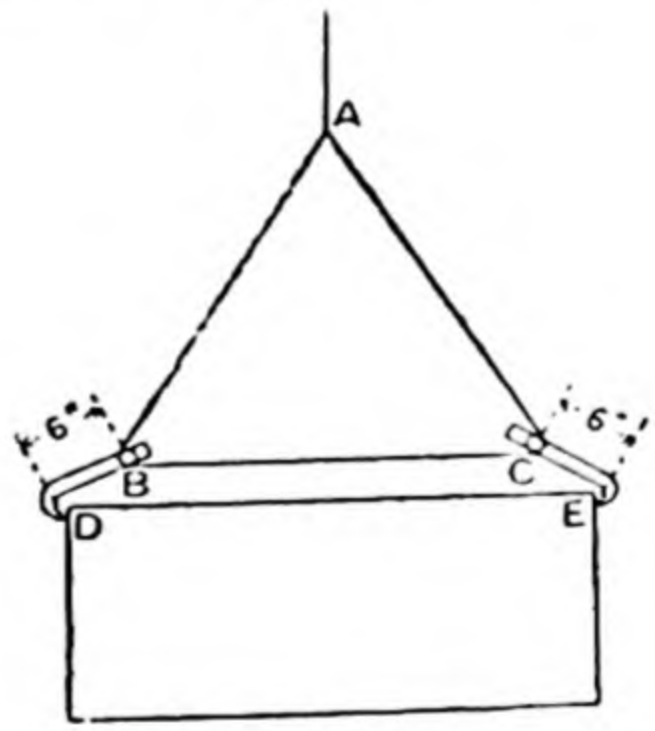



FIG. 217.

If the limiting angle of friction at D and E is 30°, and the length of the chain is 4 ft., show that the smallest block which can be lifted is about 26 in. wide. (C.U.)

**5.** A wire hangs in a vertical plane between two points in the same horizontal line, and is so nearly straight that its weight may be regarded as uniformly distributed across the span. Show that if the sag at the middle be $\frac{1}{80}$th of the span, the tension at the ends exceeds that at the middle by about $\frac{1}{8}$ %.

## S.

**1.** AB, BC, CD, DA are four rods hinged together into a rhombus. C rests on the ground, and a vertical thrust of

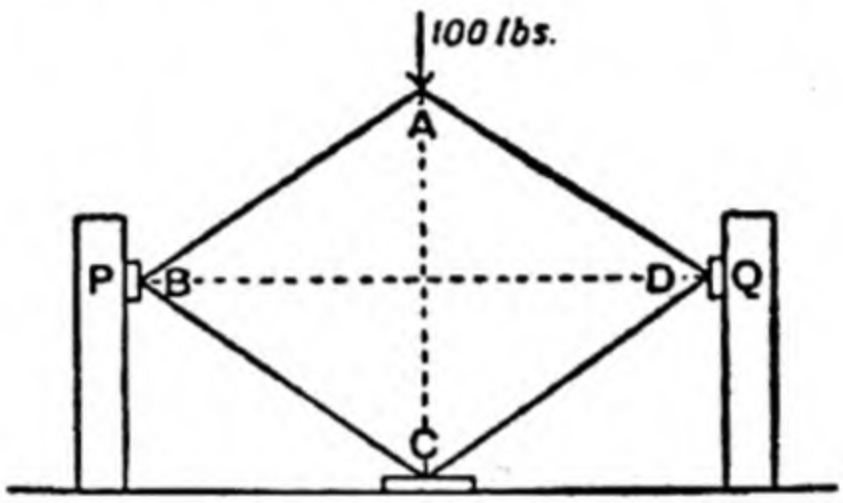



FIG. 218.

100 lbs. is applied at A. What thrust is in consequence exerted by the elbows B and D on the blocks P and Q?

$$ABC = ADC = 60°.$$ (C.S.C.)

**2.** AB is a straight horizontal beam 30 ft. long. It carries loads of 10 tons at A, 8 tons at B and 12 tons at C, which is 20 ft. from A. It is to be supported by two props, one placed 8 ft. from A and the other at such a point that the load is equally divided between the props. Find the position of the second prop by the funicular-polygon method, and verify by calculation.

**3.** A and C are fixed points, C being vertically over A. A light stiff rod AB, equal in length to AC, is pivoted at A, and a rope connects the end B to the fixed point C. A weight W hangs from B. Show that the thrust in the rod always equals W, and that when AB makes an angle $\theta$ with the horizontal the tension in the rope equals $2W \sin\left(45 - \frac{\theta}{2}\right)$. (C.S.C.)

**4.** The figure illustrates a simple type of suspension bridge, in which the roadway consists of two beams PQ, RQ freely

hinged together at Q, and supported by three vertical tie rods hung from the chain of light rods ABCDE. Prove that for the given form of the chain the tensions in the tie rods must be

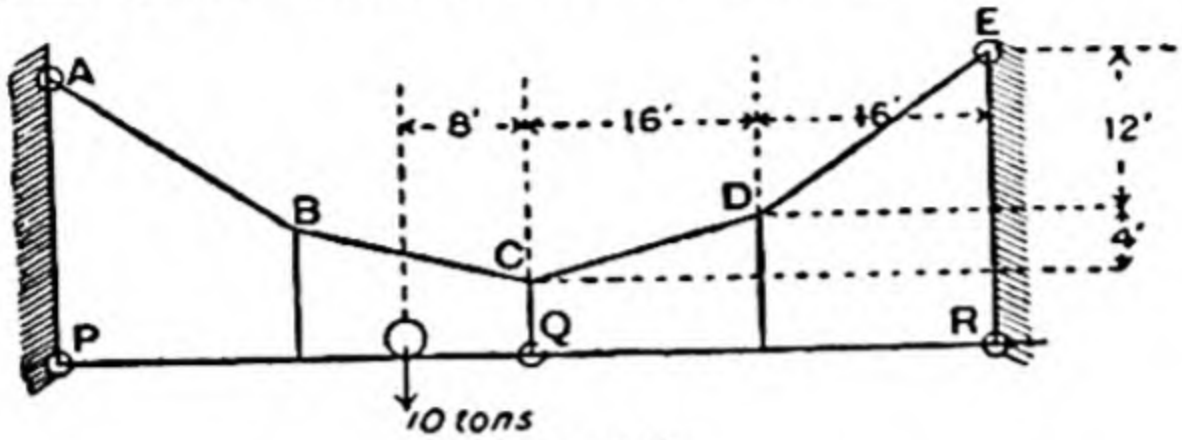



FIG. 219.

the same, and show that this tension amounts to $3\frac{3}{4}$ tons when the beam PQ is loaded in the given manner. (The weight of the beams is neglected and the load produces no distortion.) (C.U.)

**5.** Two equal masses are joined by a cord passing over a smooth pulley on the ridge of an isosceles 90° wedge; the

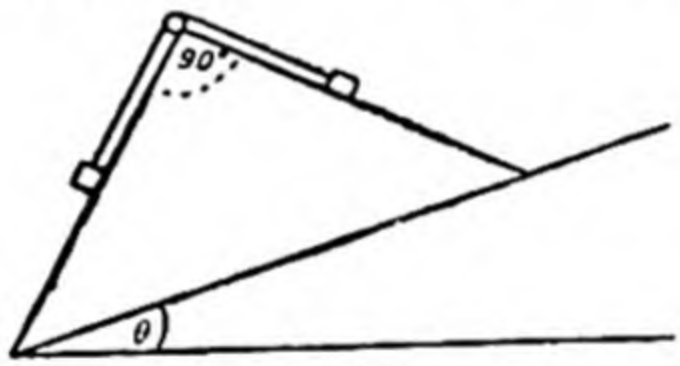



FIG. 220.

wedge is then tilted until the masses begin to move. Show that the the angle $\theta$ is then equal to the angle of frictior between the masses and the wedge. (C.U.)

## T.

**1.** A and B are fixed points. AC, CD and BD are three straight strings. AB = 6 ft., AC = 2 ft., BD = 3 ft., BAC = ABD = 45°.

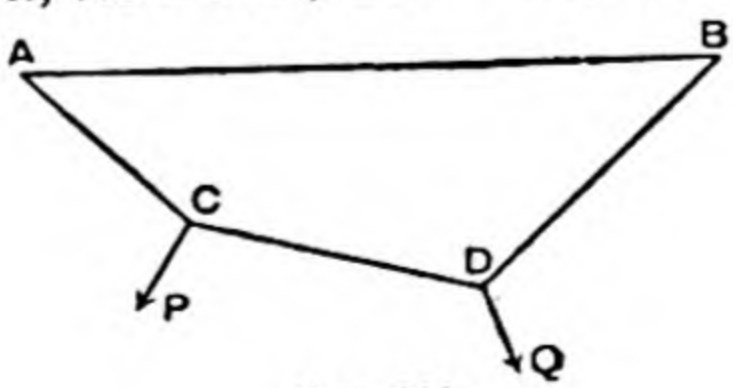



FIG. 221.

It is required to apply forces P and Q at C and D of such magnitude and in such directions that they will maintain the strings in equilibrium in the positions given, producing in each string a tension proportional to its own length. Find the magnitude and direction of these forces, if the tension in AC is to be 2 lbs. wt.

**2.** A uniform beam AB, 15 ft. long, weighing 40 lbs., is supported by a pier at C and a pier at D, where AC = 3 ft. and AD = 9 ft. Weights of 10 lbs., 14 lbs. and 16 lbs. are placed at the middle points of AC, CD and DB respectively. Find through what distance the beam and the weights on it must be moved in order that the pressures on the two piers may be equal.

**3.** In a crane the lifting force is applied to a handle which rotates in a circle of 3 ft. diameter, and it is found that when the handle makes five complete revolutions the load is lifted 1·46 ft. It is also found that with loads of 1 cwt., 3 cwt. and 5 cwt. respectively, the lifting forces required are 17·5 lbs., 26 lbs. and 35 lbs. Find the efficiency for each of these loads.

**4.** A railway wagon is placed with one pair of wheels on each of two platform weighing machines, which are on the same level, and the weights recorded are 5·4 tons on machine A and 4·6 tons on machine B. The axle over machine A is now jacked up so that it is 6 in. higher than the axle over machine B, and the distribution of the weights between the machines is now 5·1 tons on A and 4·9 tons on B. If the distance between the axles is 10 ft., show that the C. of G. of the truck is very nearly 6 ft. above the plane through the axles.

**5.** A uniform rod of length $2a$ is placed against a smooth fixed cylinder, with one end resting on a rough horizontal

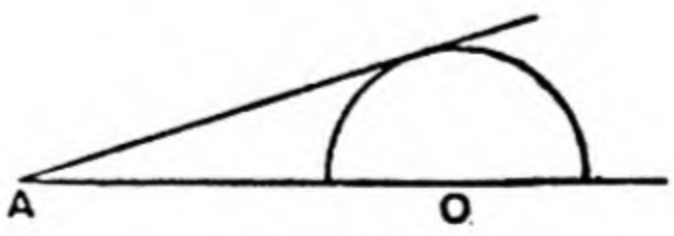



FIG. 222.

plane which passes through the centre of the cylinder. The rod makes an angle $\alpha$ with the ground, and $d$ is the distance

of the end A from the centre of the cylinder. If the rod is on the point of slipping along the plane, show that

$$\frac{P}{W} = \frac{\sin\phi}{\sin(\alpha+\phi)} = \frac{a}{d},$$

where P is the force between the rod and the cylinder and W the weight of rod, $\phi$ the angle of friction. (C.U.)

## U.

**1.** A trap-door ABC is supported by a hinge at A and a bolt at BCD. The bolt passes through rings at B, C and D. The

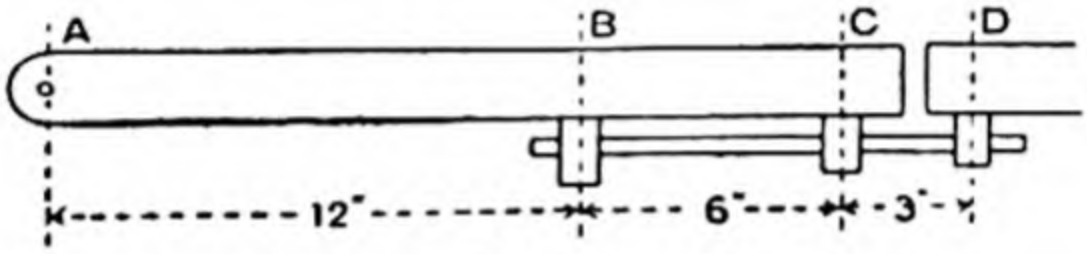



FIG. 223.

door weighs 120 lbs., and its C. of G. is 10·5 in. from A. Find the forces between the bolt and the rings at B, C and D. (C.U.)

**2.** An elastic string, whose natural length is 2 ft., is observed to stretch 4 in. when fixed at one end and supporting a mass of 2 oz. attached at the other. If this mass be now pulled aside horizontally by a force equal to the weight of 1·5 oz., find its horizontal displacement in the position of equilibrium.

**3.** Five forces in a horizontal plane acting at a point are in equilibrium. One is a force of 5 lbs. acting East, one a force of 10 lbs. acting 20° E. of N., and one a force of 8 lbs. acting 15° W. of N. The remaining two forces act S. and S.W. respectively. Find their magnitudes.

**4.** Two bodies, each of weight W, are fastened to a light rod, each 3 ft. from its mid point, and two strings are fastened to the rod, each 2 ft. from the mid point. The rod is placed on a rough horizontal table, and the strings are pulled in directions perpendicular to the rod, making the same angle $\theta$ with the vertical on opposite sides of the rod. Find the least tension that will turn the rod, and show that if $\mu = \frac{2}{3}$ the tension will be least when $\theta = 45°$.

**5.** The framework is made of smoothly jointed rods.

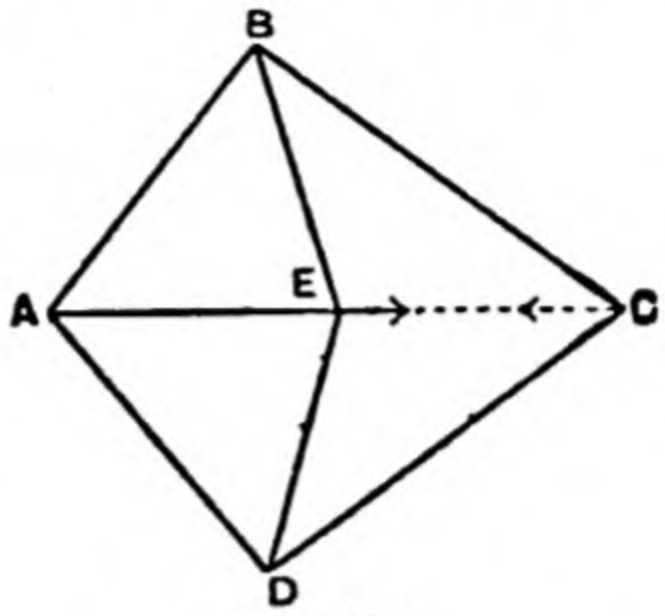



FIG. 224.

AB = AD = 3 ft., BC = DC = 4 ft., AC = 5 ft., AE = EC. Forces of 40 lbs. act at C and E as shown. Find the stresses. (C.U.)

## V.

**1.** A body of weight W is supported by two vertical strings attached to points A and B in the body. When the inclination of AB to the upward vertical is $\theta_1$ the tension in the string at A is $T_1$, and when the inclination of AB is $\theta_2$ the tension in this string is $T_2$. Prove that the perpendicular distance of the C. of G. of the body from the line AB is

$$\frac{(T_2 - T_1)\,AB}{W(\cot\theta_1 - \cot\theta_2)}. \qquad \text{(C.U.)}$$

**2.** One end of a uniform rod is placed against the slope of a smooth inclined plane of angle $\alpha$. The rod rests in a vertical plane containing a line of greatest slope of the inclined plane, and is kept at an angle $\beta$ to the horizontal by a horizontal force at its other end. Prove that $2\tan\beta = \cot\alpha$.

**3.** A wedge whose angle is $\alpha$ rests on a smooth horizontal surface, and its thin edge is inserted beneath the lower end of a vertical rod which can move vertically without friction between guides at its sides. A horizontal force P applied to the wedge makes the rod rise, so that its lower end touches the wedge along a line of greatest slope. If $\phi$ is the angle of friction between rod and wedge, find (1) the force ratio of the system, (2) its efficiency as a machine for raising the rod.

**4.** A weight of 5 lbs. rests on a board 10 ft. long, and slides down with uniform velocity when one end of the board is raised a vertical height of 2 ft. above the other end. What force parallel to the board would be required to keep the weight from moving if the end were raised another foot ?

**5.** AB and CD are horizontal. The frame is supported at

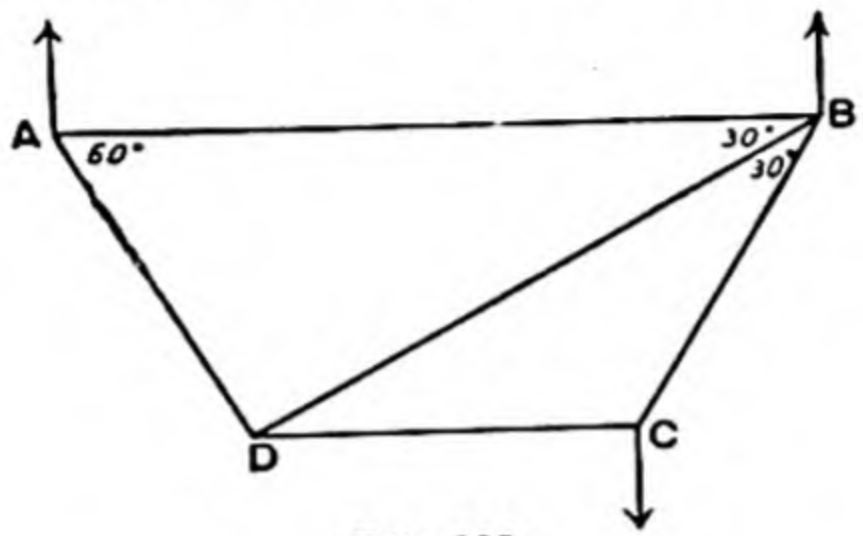



FIG. 225.

A and B, and carries a load of 1000 lbs. at C. Find the supporting forces at A and B and the forces in the rods.

W.

**1.** A smooth book of weight W, supposed to be a uniform rectangular block of height $a$ and thickness $b$, leans over at an angle $\alpha$ to the vertical, between vertical books on each side of it on a smooth shelf. Prove that it presses against these books with a force

$$\frac{W}{2}\,\frac{a\sin\alpha - b\cos\alpha}{a\cos\alpha - b\sin\alpha}.$$

**2.** A span AD is bridged by two beams joined together as shown, the upper beam resting on the lower at B and the lower

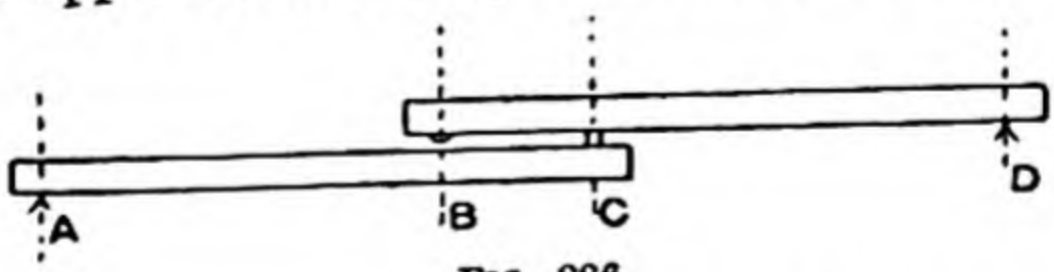



FIG. 226.

hanging from the upper by a bolt at C. Show how the tension in the bolt varies as a load moves across the bridge from A to D, and prove that the tension is greatest when the load is over B.

If the span is 30 ft. and the points B and C each 3 ft. from the middle of the span, find the greatest value of the tension in the bolt when a man weighing 13 stones walks across the bridge. (C.U.)

**3.** A uniform rod AB, 4 ft. long, weighing 10 lbs., can turn freely about a hinge at A. A rough heavy ring C can slide on the rod, and is fastened to a fixed point D by a light string.

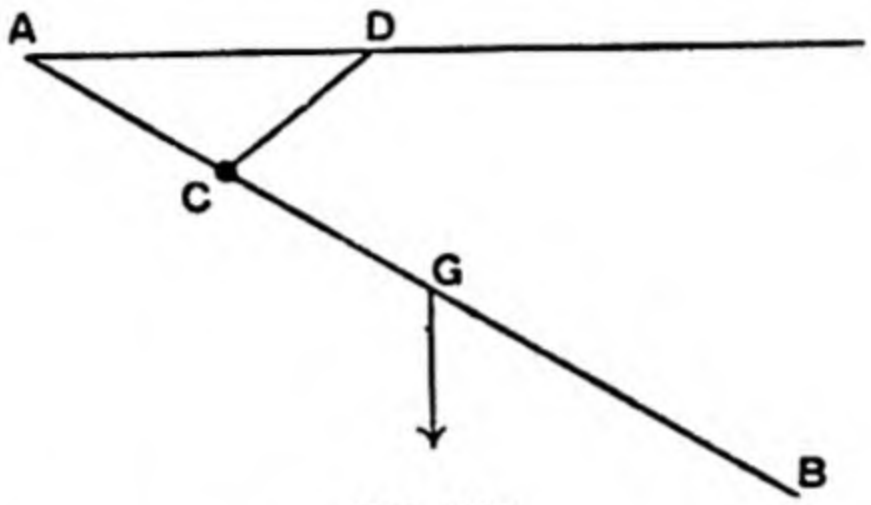



FIG. 227.

AC = 1 ft., AD = $1\frac{1}{4}$ ft., DAC = 30°. Find the weight of the ring if it is on the point of slipping down the rod in the given position, $\mu$ for rod and ring being $\frac{1}{4}$.

**4.** A light beam rests on supports 15 ft. apart, and carries loads of 7, 4, 2 tons at distances 3, 8, 12 ft. respectively from the left-hand end. Calculate the B.M. at 4, 9 and 10 ft. from the same end.

**5.** The sketch shows a column D, of weight W, which is being raised vertically, with uniform velocity, by a force P

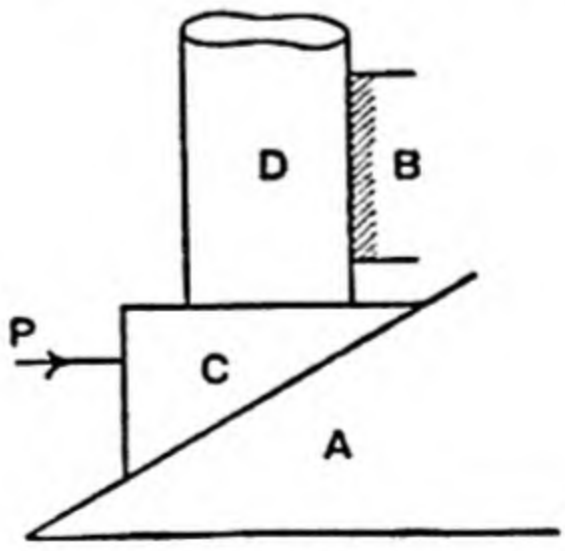



FIG. 228.

applied to the wedge C. The inclined plane A and the support B are fixed, and the upper face of the wedge is horizontal.

There is the same coefficient of friction between A and C as between C and D and D and B. Draw a diagram of the forces acting on the wedge and on the pillar, and find the ratio of P to W, when $\mu = \tan 18°$ and the inclination of the plane is 30°.
(C.U.)

## X.

**1.** In the crane it is found that when W is at rest and of magnitude $\frac{1}{2}$ ton, the resultant pressure on the ground acts in

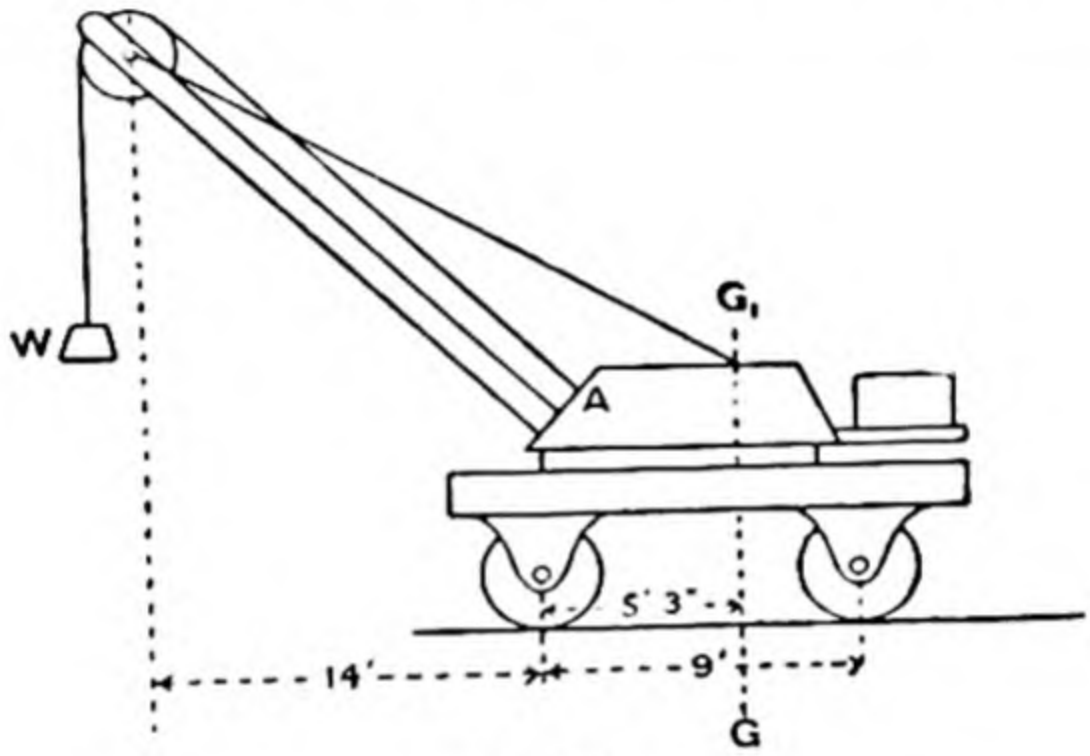



FIG. 229.

the line $G_1G$ and is 6 tons wt. Determine the pressures upon the ground when W is 2 tons, and the load is being raised with uniform velocity. The rope is wound on to a drum at A.
(C.U.)

**2.** Two rings of weight 2 lbs. and 3 lbs. respectively slide on a string whose ends are fastened to the extremities of a straight rod inclined at $\theta°$ to the horizon ; on the rod slides a light ring, through which the string passes, so that the heavy rings are on different sides of the light ring. Prove that in the position of equilibrium the inclination $\phi$ of the rod, to those parts of the string next the weightless ring, is given by $\tan \phi = 5 \tan \theta$, assuming that friction is negligible.

**3.** A double wedge has as its section the triangle ABC, where $A = 30°$, $B = 90°$ and the wedge is held fixed with AC horizontal. A weight of 10 lbs. resting on AB and a weight W resting on BC

are connected by a string passing over a smooth pulley at B. Find the greatest value W can have consistent with equilibrium, if the coefficient of friction between each weight and the plane on which it rests is $\frac{3}{4}$.

**4.** A uniform rectangular board ABCD has hinges attached at two points in the edge AB, so that it can turn about AB as a horizontal axis. Strings are attached at C and D, which pass over pulleys in the same horizontal plane as AB, and support equal masses which hang freely. The distance of the pulleys from the edge AB is equal to the length of the edge AD. Find an expression for the inclination of the board in terms of the weights of the board and the hanging masses; and show that if the weight of the board is less than twice the sum of the weights at the ends of the strings, the only possible position of equilibrium for the board is the horizontal position.

Determine the position of equilibrium of the board when its weight is 10 lbs., and each string supports a weight of 2 lbs. (C.S.C.)

**5.** ABC is a light bent lever, which can turn in a vertical plane about A. It supports at C a uniform rod ED weighing

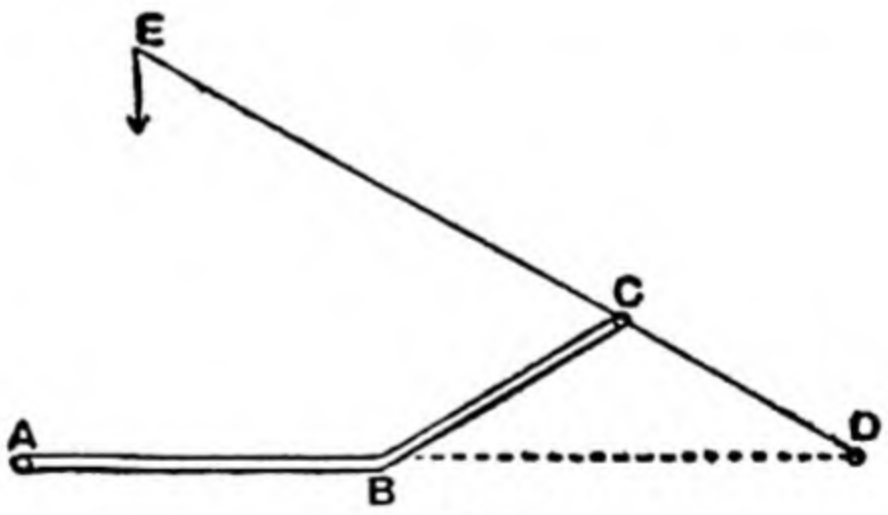



FIG. 220 (*a*).

2 lbs., which can turn about D. $A\hat{B}C = 150°$, BC = CD = 1 ft., CE = 2 ft. Find what vertical force at B is required to keep AB horizontal when 20 lbs. is hung at E.

## Y.

**1.** The beam AD of the given lever turns about C. Vertical rods BE, DF are attached to the horizontal levers EG, FH; EG

being pivoted to FH at G. A weight at A keeps the levers horizontal. Prove that if a load W is placed on EG, the addi-

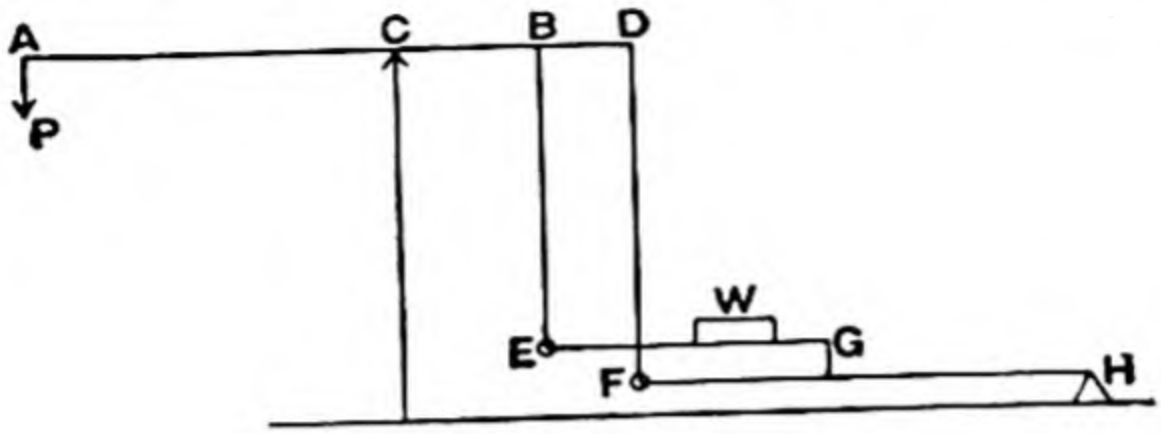



FIG. 229 (*b*).

tional weight P required at A is given by the relation

$$P(AC) = W(BC), \text{ provided } \frac{CD}{BC} = \frac{FH}{GH};$$

this result being independent of the position of W on EG.

**2.** The light rods AB, BC, CD are freely hinged at B. BC can turn about a fixed pivot at C. The hammer ED is con-

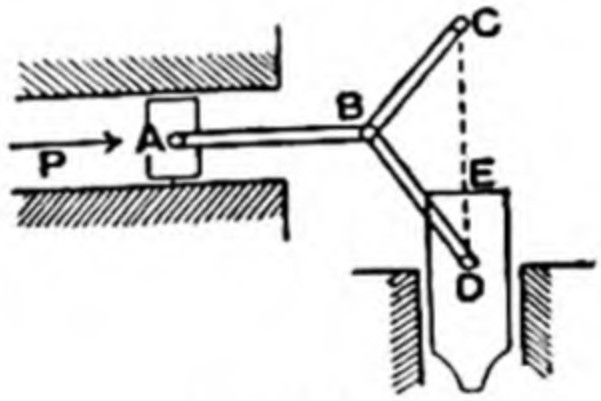



FIG. 229 (*c*).

strained to move vertically by smooth guides. If BD = 6 in., BC = 3 in., $D\hat{C}B = 20°$, find what vertical thrust along ED is produced on the hammer when a horizontal force P = 1000 lbs. acts on A.

**3.** A uniform rod AB, 3 ft. long, weighing 10 lbs., can turn freely about B and is held at 30° to the horizontal with A below B, by a string fastened to A and inclined at 45° to the vertical on the other side; the string passes over a fixed rough circular cylinder of radius 1 ft., and supports a weight W at its other end. Find the value of W when it is about to move upwards, given that the tension T in the string between

the cylinder and the rod is $We^{0\cdot2\theta}$, where $\theta$ is the angle in radians subtended at the centre by the string in contact with the cylinder. ($e$ is the base of Naperian logarithms $=2\cdot72$ approximately.)

**4.** A cone of vertical angle 30° and weight W is placed with its vertex on a smooth horizontal plane. Prove that the couple necessary to hold it with its slant side vertical is $\frac{3Wl}{16}$, where $l$ is the length of the slant side.

**5.** The weight of a ship and its cargo is 4000 tons. If 100 tons of cargo is moved 20 ft. transversely across the deck, find the consequent displacement of the C. of G. of the total load, assuming that the deck remains horizontal.

## Z.

In the following questions, part of a solution is given. Discuss whether the solution is correct or incorrect. If wrong, point out the mistake and give the correct solution.

**1.** A weight of 5 lbs. is fastened to a smooth light ring C, which runs on a string ACB. The string is fastened to A, passes over a small pulley at B on the same level as A and supports a weight W. If $ACB=60°$, find W.

*Solution.* Let T be the tension in the string. Since the ring is smooth, AC and CB will each be inclined at 30° to the vertical.

$\therefore$ resolving vertically, $\quad 2T\cos 30°=5.$

Also, at B we have $\quad T\cos 30°=W;$

$$\therefore\ 2W=5;\quad \therefore\ W=2\cdot5 \text{ lbs.}$$

**2.** A cubical block of stone (edge 1 ft.) weighs 200 lbs. and rests on the ground. It is just lifted by a uniform lever 3 ft. long, weighing 1 lb., which is pushed 3 in. under the block at right angles to an edge of the base and pulled vertically at the other end. If the lever is considered to be horizontal, find P.

*Solution.* Take moments about the edge of the block.

$$P(33)-1(15)=200(6);\quad \therefore\ P=\tfrac{1200}{33}\tfrac{15}{3}=36\tfrac{9}{11} \text{ lbs.}$$

**3.** A weight of 12 lbs. rests on an inclined plane, and moves down slowly with uniform velocity, when the plane makes 20° with the horizontal. What force parallel to the plane would

be required to move the weight slowly up, if the plane were inclined at 30° to the horizontal ?

*Solution.* Resolving along the plane,

$$F = 12 \sin 20, \text{ where } F \text{ is the friction.}$$

If the plane were inclined at 30°, let P be the required force : then

$$P = 12 \sin 30 + F = 12 \sin 30 + 12 \sin 20$$
$$= 12 \sin 50 \quad = 12(\cdot 766) = 9 \cdot 2 \text{ lbs.}$$

**4.** A body weighing 20 lbs. is dragged along a rough horizontal plane by a force inclined at 40° to the plane. Find the least value of the force when $\mu = 0 \cdot 2$.

*Solution.* Let P be the required force, N the pressure on the plane. Then friction $= (0 \cdot 2)N = (0 \cdot 2)(20) = 4$ lbs.

Resolving horizontally,

$$P \cos 40 = 4 ; \quad \therefore \ P = 4 \sec 40 = 5 \cdot 22 \text{ lbs.}$$

**5.** Find the C. of G. of an irregular quadrilateral.

*Solution.* The C. of G. of a quadrilateral coincides with the C. of G. of four equal weights placed at the angular points A, B, C, D.

The C. of G. of the weights at A and C will be at the mid point of AC. Similarly for BD.

Therefore the C. of G. of the quadrilateral is the mid point of the line joining the mid points of the two diagonals.

# CHAPTER IX.

## GENERAL CONDITIONS OF EQUILIBRIUM.

A FORCE acting on a rigid body at a given point may be transferred to any other point, provided a couple is also introduced to act upon the body.

Let the force P act on the rigid body at A.

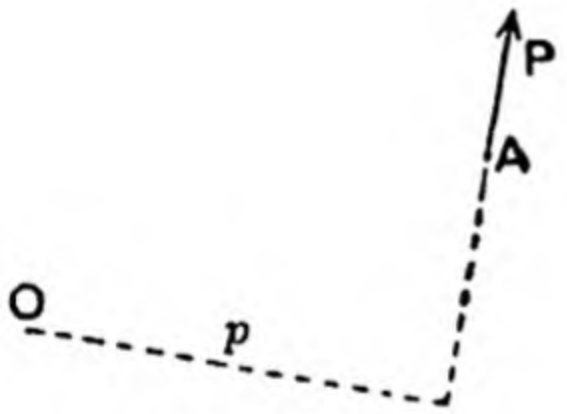



FIG. 230 (*a*).

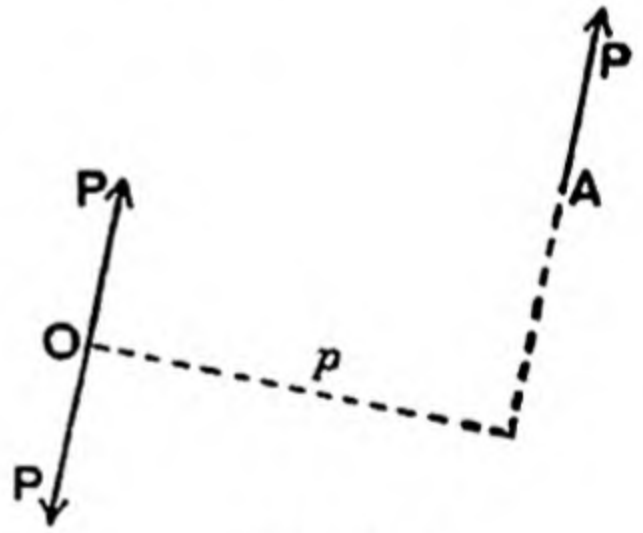



FIG. 230 (*b*).

It is clear that the force P acting at A tends to turn the body round O, and that its moment about O = P$p$.

At O, any other point, introduce two equal and opposite forces P, parallel to the force P at A. These will not affect the equilibrium of the body. We may now consider that the body is acted upon by a force P at O, together with a couple P$p$.

The moment about O of P acting at O is zero, and the moment of the couple is P$p$.

**To find the Conditions of Equilibrium for a Rigid Body acted upon by a Number of Forces at various Points in a Plane.**

Each force P, Q, ... acting at points A, B, ... may be transferred to act at any point O, provided we introduce a system of couples whose moments will be P$p$, Q$q$, etc.

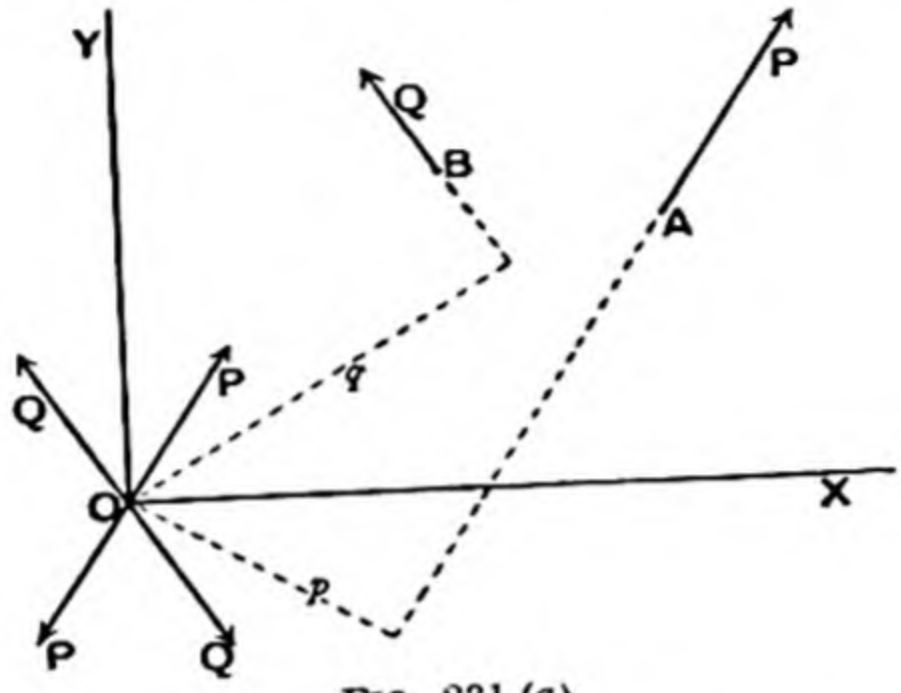



FIG. 231 (*a*).

Through O draw two axes at right angles to one another.

The forces P, Q, ... at O may be resolved in the two directions OX, OY.

Let the algebraical sum of these components be X, Y respectively.

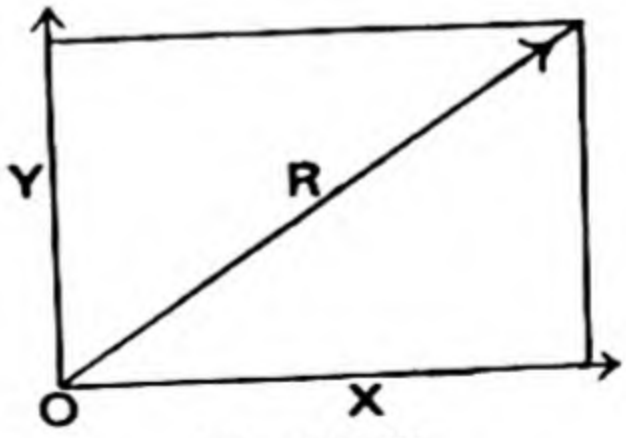



FIG. 231 (*b*).

The resultant of these will be R, where $R^2 = X^2 + Y^2$.

The moment of the resultant couple is obtained by

finding the algebraical sum of the couples $Pp$, $Qq$, .... Let the moment of this resultant couple be C.

The resultant force cannot balance the resultant couple, therefore for equilibrium there must be no resultant force R, and there must be no resultant couple;

$$\therefore X=0,\ Y=0 \text{ and } C=0.$$

It will now be clear that in practice we need not go through the operation of transferring the forces to a point O, since the resolved parts may be obtained by resolving the forces at the points A, B, etc., where they act.

The moment of each couple is independent of the position of the point about which we take moments. Now the moment of the couple $Pp$ is the same as the moment of the force P (acting at A) about the point O. ∴ the resultant couple will vanish if the algebraical sum of the moments of the various forces about any point O is zero.

*Notes.*—I. The equations we have obtained amount to the fact that if $X=0$, the body will not move in the direction of the X-axis. If also $Y=0$, neither can it move in the direction of the Y-axis. If $C=0$, it cannot rotate; therefore it is in equilibrium.

II. If $X=0$ and $Y=0$, the resolved parts of the forces *in any direction* must be zero. The original forces being

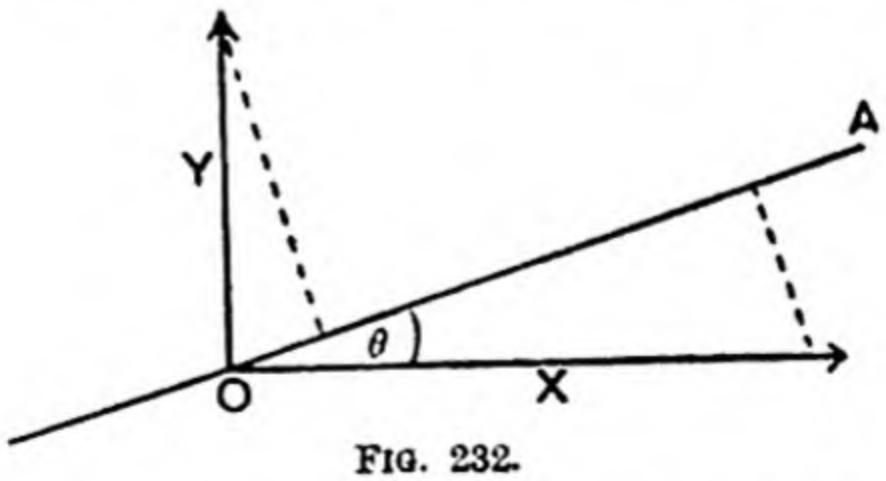



FIG. 232.

equivalent to X and Y, we have along OA the force

$$X\cos\theta + Y\sin\theta = 0.$$

III. If the sum of the moments about a certain point, of a set of forces, is zero, it follows that the forces are not equivalent to a couple.

This condition holds good (i) if the resultant of the forces is zero, or (ii) if the certain point is on the line of action of the resultant.

IV. If the sum of the moments of a set of forces about three non-collinear points each separately equals zero, the forces must be in equilibrium.

If there is a resultant, its moment about each of the three points must be zero, but, as the points are not collinear, the resultant cannot pass through all three, ∴ there is no resultant; and as there is no resultant couple, the forces must be in equilibrium.

EXAMPLE 1. A uniform rod AB, weighing 10 lbs., is hinged at A and is supported by a force P as in the diagram.

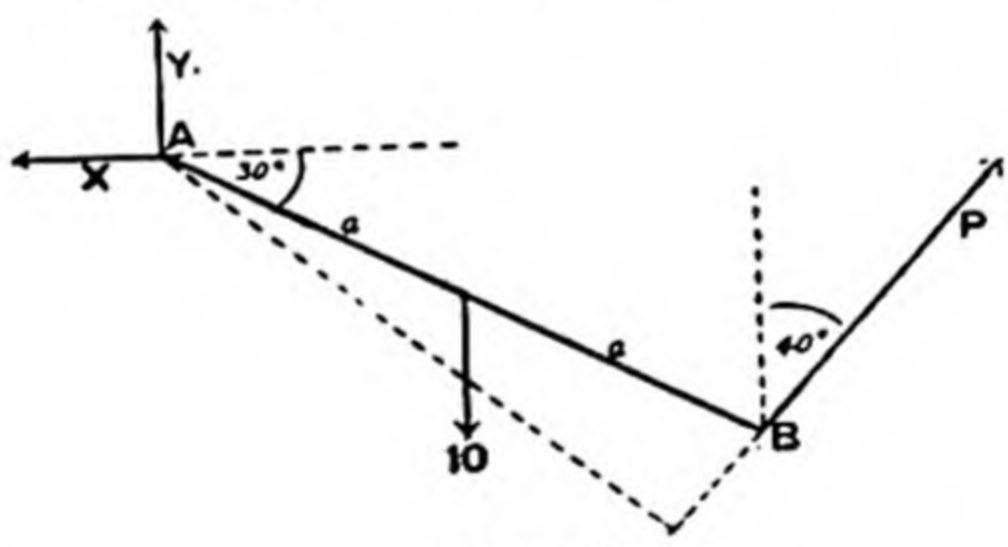



FIG. 233.

To find the reaction at the hinge, *i.e.* the force exerted by the pin at the hinge on the rod.

Let this force produced by the pin be resolved horizontally and vertically into two components X and Y. (This reaction could, of course, easily be found by using the triangle of forces.)

Resolving all the forces on the rod horizontally, we have, since the rod is in equilibrium,

$$X - P \sin 40 = 0.$$

Vertically, $Y + P\cos 40 - 10 = 0.$

The moment of the resultant couple (by taking moments about A) is $10a\cos 30 - P2a\sin 80 = 0.$

From which we find

$$P = 4{\cdot}4 \text{ lbs.}, \quad X = 2{\cdot}8 \text{ lbs.}, \quad Y = 6{\cdot}6 \text{ lbs};$$

$\therefore$ the reaction at the hinge is $\sqrt{X^2 + Y^2} = 7{\cdot}1$ lbs.

EXAMPLE 2. Two equal uniform rods, each weighing 20 lbs., are hinged at A, and their ends B and C rest on a smooth

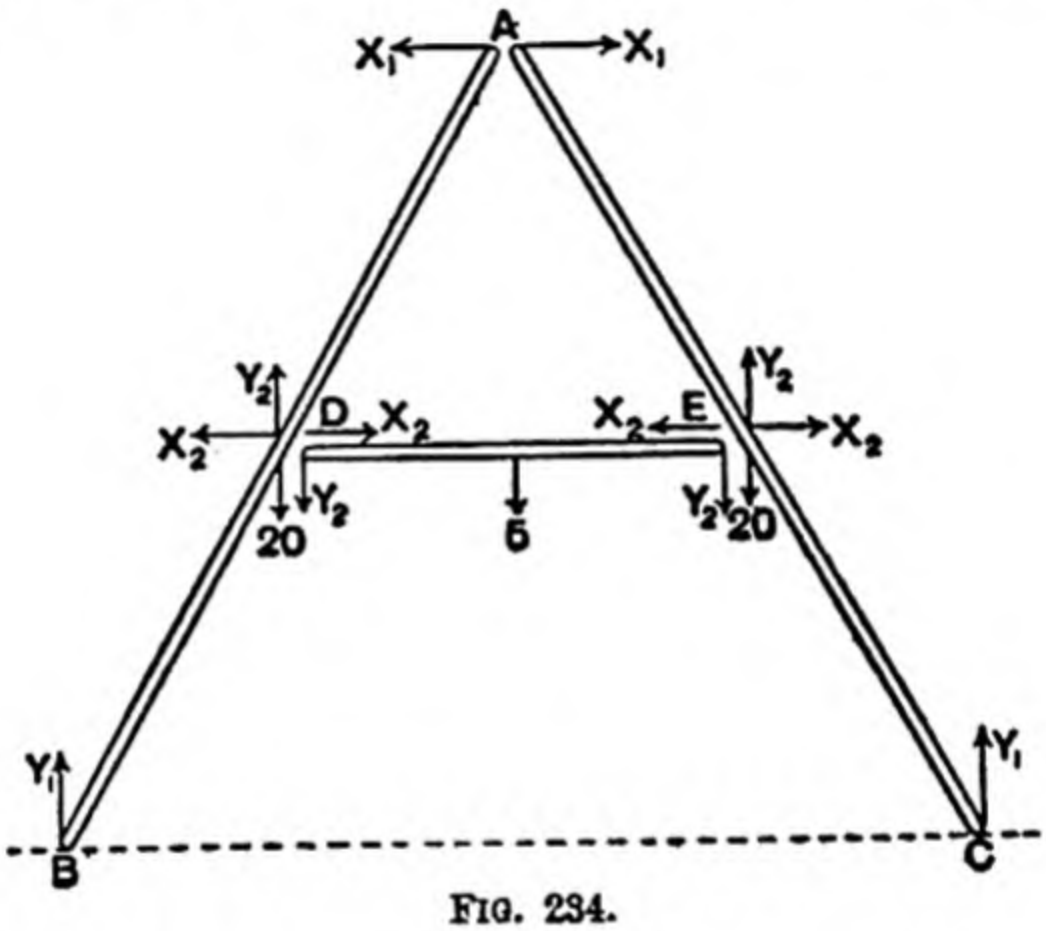



FIG. 234.

horizontal floor. A third rod DE, of weight 5 lbs., hinged to AB and AC at their mid points, prevents them from opening out. Find the reactions of the hinges when $BAC = 60°$.

The action of the hinge at A on AB must be equal and opposite to its action on AC. It must therefore be entirely horizontal, for if it had a vertical component, this force would act upwards on one rod and downwards on the other; but as the figure is symmetrical, the action at A on both rods must be symmetrical; $\therefore$ there can be no vertical component. Let the components of the reaction of the hinges at D and E be $X_2$, $Y_2$; these forces will act in opposite directions on the rods AB and DE, AC and DE.

It is obvious that the horizontal forces ($X_2$) on the rod DE will act outwards, and that the vertical forces ($Y_2$) will act

upwards; but if we put them incorrectly on the diagram, they will appear with a negative sign when the equations are solved.

The reactions at the ground will be vertical.

Taking all the rods together as one system, the reactions at the hinges will disappear, and we have, resolving vertically,

$$2Y_1 = 45 \; ; \quad \therefore \; Y_1 = 22{\cdot}5.$$

Taking the rod DE, and resolving vertically, we have

$$2Y_2 + 5 = 0 \; ; \quad \therefore \; Y_2 = -2{\cdot}5.$$

This shows that the direction of $Y_2$ should be upwards on DE and downwards on the sloping rods.

Consider the equilibrium of AB alone.

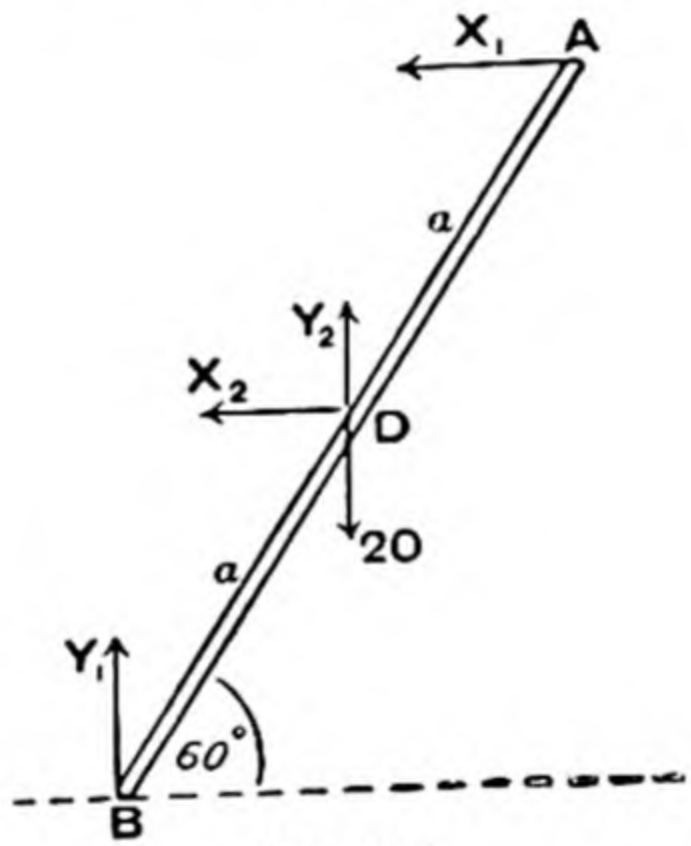



FIG. 235.

Resolving horizontally,

$$X_1 + X_2 = 0 \; ; \quad \therefore \; X_1 = -X_2.$$

Vertically,

$$Y_2 + Y_1 = 20.$$

(This we have already obtained.)

Taking moments about the mid point,

$$X_1 a \sin 60 = Y_1 a \cos 60 \; ;$$

$$\therefore \; X_1 = Y_1 \cot 60 = \frac{(22{\cdot}5)}{\sqrt{3}} = 13 = -X_2;$$

$\therefore$ reaction at D $= \sqrt{X_2^2 + Y_2^2} = \sqrt{13^2 + 2 \cdot 5^2} = 13 \cdot 25$ lbs.

Reaction at A $= 13$ lbs.

Reaction at B $= 22 \cdot 5$ lbs.

EXAMPLE 3. A hollow cylinder, radius 1 inch, weighing 4 oz., is open at both ends, and is placed on a smooth horizontal plane. Two equal balls, radius $\frac{3}{4}''$, are then placed inside the cylinder. Find the weight of these balls, if the cylinder will just not upset.

Since the cylinder is about to upset, the reaction of the ground will be entirely at C.

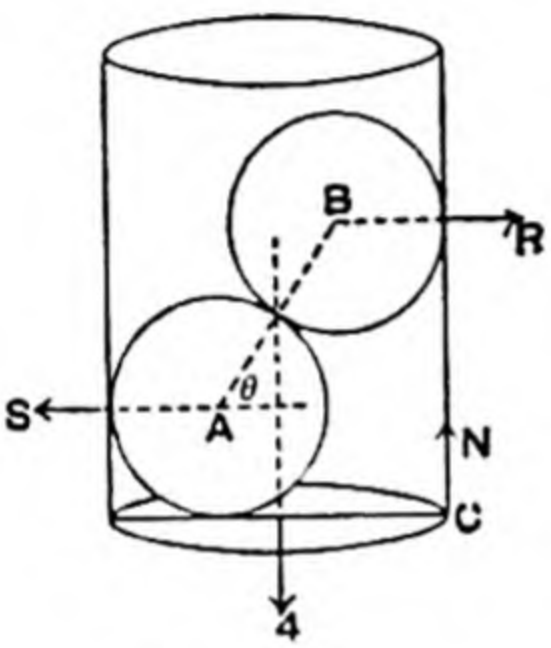



FIG. 236 (*a*).

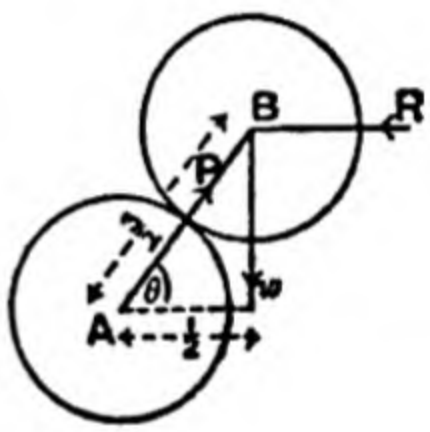



FIG. 236 (*b*).

The forces *on the cylinder* are :

R due to the upper ball, S due to the lower ball.

4 oz. the weight of the cylinder and N the vertical reaction at C.

Resolving horizontally, $R - S = 0$ ; $\therefore R = S$.

,, vertically, $N - 4 = 0$ ; $\therefore N = 4$.

Taking moments about C,

$$R \cdot (\tfrac{3}{4} + \tfrac{3}{2} \sin\theta) - S \cdot \tfrac{3}{4} = 4 \cdot 1 ;$$

$$\therefore R \cdot \tfrac{3}{2} \sin\theta = 4 ;$$

$$\therefore R = \frac{8}{3 \sin\theta}.$$

Consider the forces acting on the upper sphere.

Resolving horizontally, $R = P \cos\theta$.

Vertically, $$w = \mathrm{P} \sin\theta.$$

$$\therefore \tan\theta = \frac{w}{\mathrm{R}};$$

$$\therefore w = \mathrm{R}\tan\theta = \tfrac{8}{3}\sec\theta = 8 \text{ oz.}$$

EXAMPLE 4. *Motion of a Bicycle.*—The pressure P on the pedal produces a torque acting on the chain wheel equal to

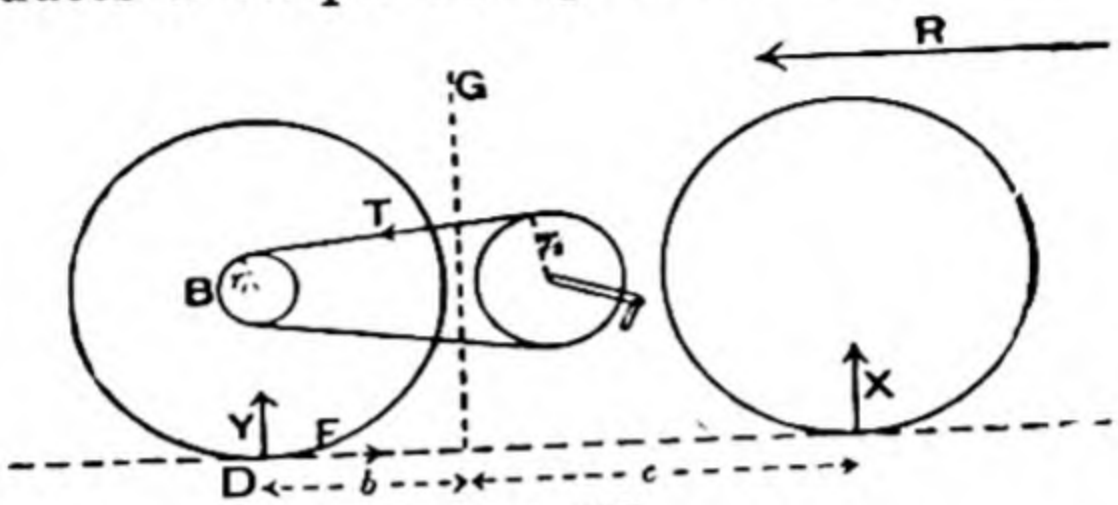



FIG. 237.

the torque produced by the tension in the chain. The tension T in the chain produces a torque on the driving wheel (the lower part of the chain is slack), and when the bicycle is moving with uniform velocity, there must be an opposing couple of equal torque acting on the rear wheel. This is produced by the friction of the ground. (If the ground were smooth, the wheel would rotate without advancing.)

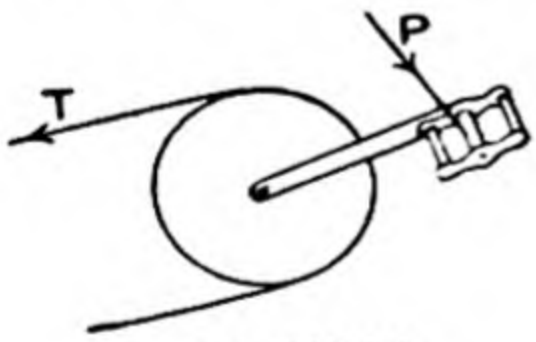



FIG. 238 (*a*).

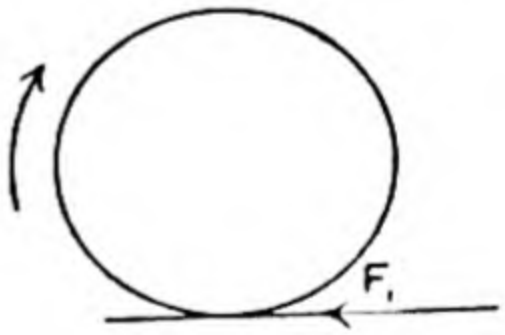



FIG. 238 (*b*).

The force F keeps the point D instantaneously at rest, and is the only *external* force acting on the bicycle to produce a forward motion.

The front wheel is made to rotate by the friction $F_1$ at its point of contact with the ground, but as this force has only to overcome the friction at the axle and is small, we shall neglect it. (Fig. 238(*b*).)

We have then, if $r_1$ is the radius of the hub, $\rho$ the radius of the back wheel, $Tr_1 = F\rho$ (taking moments about the axle, Fig. 237).

Similarly for the chain wheel, $Pp = Tr_2$, where $p$ is the length of the crank. (Fig. 238 (*a*).)

$$\therefore\ P = \frac{F\rho r_2}{r_1 p}.$$

Now $r_1$ revolutions of the chain wheel $= r_2$ revolutions of the back wheel ; $\therefore$ if $g$ is the gear of the bicycle (see p. 66, Ex. 12),

$$\frac{r_2}{r_1} . 2\pi\rho = \pi g\ ;\quad \therefore\ P = \frac{Fg}{2p} = \frac{Rg}{2p},\ \text{since } F = R.$$

If the action of P is considered to be entirely vertical, we have, by the principle of work for one revolution of the pedal,

$$2P . 2p = R\pi g\ ;\quad \therefore\ P = \frac{\pi g R}{4p}.$$

Supposing the resultant air resistance to act through G, we have, by resolving for the whole bicycle,

Horizontally, $R = F$. .....(1) Vertically, $X + Y = W$. .....(2)

Taking moments about G, if $h$ is the distance of G from the ground, $Fh + Xc = Yb\ ;\quad \therefore\ Rh = Yb - Xc$. .....(3)

Hence, from (2) and (3), we get

$$Y = \frac{Wc + Rh}{c + b},\qquad X = \frac{Wb - Rh}{c + b}.$$

## EXAMPLES XXX.

**1.** The rectangle ABCD represents a vertical section through the C. of G of a board which rests in the angle between a smooth wall and floor, with A in contact with the wall and B on the floor. $AB = 20''$, $BC = 40''$. A horizontal string at D, fastened to the wall, maintains equilibrium. If the board weighs 10 lbs., find the tension in the string when BC makes $50°$ with the horizontal.

**2.** A gate, weighing 150 lbs., is supported by two hinges 3 ft. apart, in such a way that the lower hinge takes all the weight. If the vertical through the C. of G. is $2\frac{1}{2}$ ft. from the line of the hinges, find the reactions at the hinges. What will these reactions become when a boy of 10 stone sits on the gate 4 feet from the end that is hinged ?

**3.** A uniform rod AB, 4 ft. long, weighing 4 lbs., can turn freely in a vertical plane about A. A weight of 5 lbs. is

attached to B, and the rod is kept at 40° to the horizontal, with B below A, by a horizontal force P applied 1 foot from B. Find the tension in this string and the reaction at A.

**4.** Two uniform beams AB, AC, each 6 ft. long, are jointed at A and placed in a vertical plane with B and C on a smooth floor. A string 4 ft. long joins the mid points of the beams. Find the tension in the string and the reaction at A, if each beam weighs 20 lbs.

**5.** Two light rods AB, AC, each 4 ft. long, are hinged at A and placed in a vertical plane, with the ends B and C resting on a smooth horizontal plane. B and C are joined by a string 4 ft. long, and a weight W is fastened to the rod AC at a point D, $h$ ft. above the horizontal plane. Prove that the tension in the string is $\frac{Wh}{12}$. (C.S.C.)

**6.** Three uniform rods AB, BC, CD, each 1 ft. long, are jointed at B and C, and are placed over two pegs in the same horizontal line in such a way that the three rods are horizontal. If each rod weighs 1 lb., find how far apart the pegs must be placed, and find the reaction at the hinges.

**7.** Two uniform rods AB, BC, each weighing 1 lb., are jointed at B. AB is able to move freely in a vertical plane round A. Find where a vertical force P must act on BC so that both rods may be horizontal.

**8.** A carrier ABC is supported on a horizontal rail DE by the pegs A and B, and can slide along the rail. There is

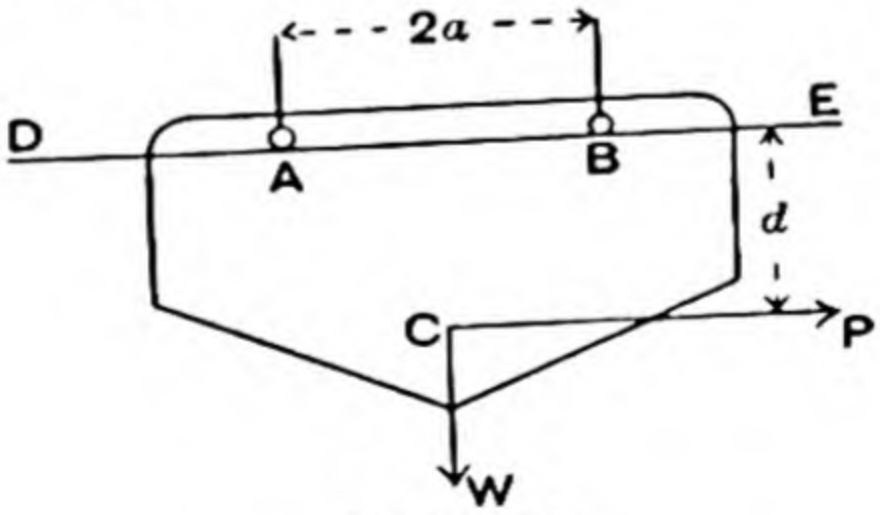



FIG. 239 (a).

friction between the pegs and the rail (coefficient $\mu$). The carrier supports a load W, and just moves when a force P

is applied horizontally at a point C vertically below the mid point of AB. AB=2$a$ and the distance of C below the top rail is $d\left(d<\frac{a}{\mu}\right)$. Find the values of P, and of $R_1$ and $R_2$ the vertical forces between the pegs A and B and the rail. Show that $\frac{R_1}{R_2}=\frac{a+\mu d}{a-\mu d}$. (C.S.C.)

**9.** A vertical rod, 3 ft. long, weighing 2 lbs., passes between two rings and rests with its lower end on a smooth plane inclined at 45° to the horizontal. If the rings are 1 ft. and 2 ft. from the lower end of the rod, find the pressure on each ring.

**10.** AC, CB are two uniform beams of equal length, and of weights $w$ and $w'$ inclined to the horizontal at an angle $\alpha$ and pin-jointed at C, the ends A and B being hinged at two fixed points at the same level. Find the pressure between the two beams at C, showing that its horizontal and vertical components are respectively $\frac{1}{4}(w+w')\cot\alpha$ and $\frac{1}{4}(w-w')$.

**11.** Two uniform rods AB and CD, each of weight W and length $a$, are smoothly jointed together at a point O, where OB and OD are each of length $b$. The rods rest in a vertical plane with the ends A and C on a smooth table and the ends B and D connected by a light string. Prove that the reaction at the joint is $\frac{aW}{2b}\tan\alpha$, where $\alpha$ is the inclination of either rod to the vertical.

**12.** A uniform plank AB, 12 ft. long, weighing 80 lbs., rests with A against a smooth wall and B on a smooth floor, at 50° to the horizontal. A rope tied 3 ft. from the bottom of the plank is fastened to the bottom of the wall and prevents the plank from slipping. Find the tension in the rope.

**13.** Two equal frames AOC, BOD are supported by hinges at C and D, and are connected by a smooth joint at O. AOB is horizontal, AC and BD are vertical. AB=100 ft. O is 30 ft. above CD. The weight of each frame is 50 tons, and the distance between their C. of G. is 80 ft. When W (20 tons) is

placed on AO, $p$ ft. from A, the horizontal and vertical forces exerted by BOD on AOC at O are X towards A and Y upwards. Prove that $3X - 5Y = 50$, and find X and Y in terms of $p$. (C.U.)

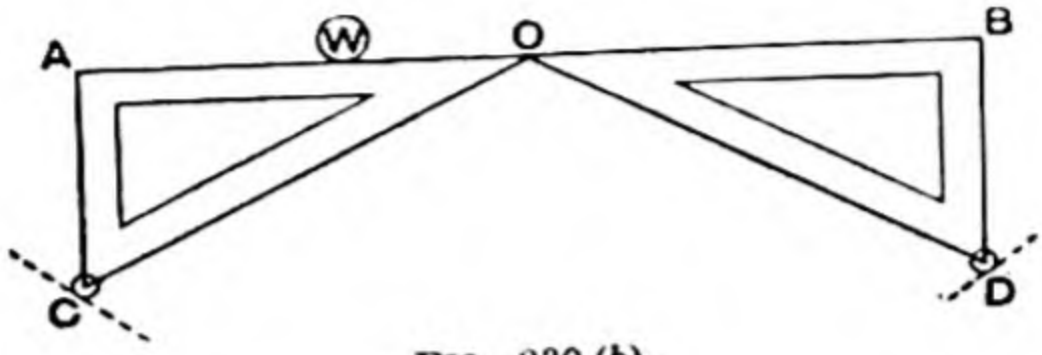



FIG. 239 (*b*).

**14.** Three equal uniform rods AB, BC, CD, each 2 ft. long, are jointed at B and C and placed symmetrically over two smooth pegs which are 3 ft. apart. Find the reactions at B and C if each rod weighs 2 lbs.

**15.** Three equal rods AB, BC, CA are jointed at A, B and C, and the triangular framework is placed over a peg at the mid point of AB, so that C is below the peg. If each rod weighs 2 lbs., find the reaction at the joints.

**16.** Six equal cylinders are placed on a smooth horizontal plane with their axes parallel, three in the lowest row, then two and one above. If each weighs 10 lbs., find the least horizontal force which must be applied to the two outside bottom cylinders to prevent collapse, and find the pressures between the cylinders when that force is applied.

**17.** The slide D can move without friction along the light rod AB. Another light rod is hinged to D and to the wall

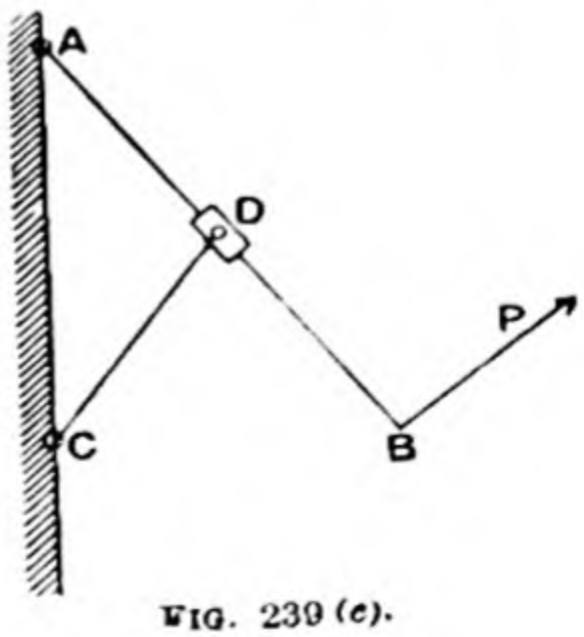



FIG. 239 (*c*).

at C. If the weight of the slide is W, prove that when P, acting at right angles to AB, maintains equilibrium,

$$\frac{P}{W}=\frac{(AD)(DE)}{(DF)(AB)};$$

DE being the perpendicular from D on AC and CF the perpendicular from C on AB.

**18.** Three uniform rods AB, BC, CD, whose weights are 1 lb., 2 lbs., 1 lb. and lengths 1 ft., 2 ft., 1 ft. respectively, are jointed at B and C and placed on a smooth cylinder, so that BC is horizontal and the plane of the rods at right angles to the axis of the cylinder. If the radius of the cylinder is $1\frac{1}{2}$ ft., find the reaction at the joints.

**19.** AB, CD are two equal uniform rods, which can turn freely about A and C respectively, AC being horizontal. To B and D are fastened the ends of a weightless string, which passes through a ring R, whose weight is half that of either of the rods AB, CD. Show that if in the position of equilibrium AB and BR make angles $\alpha$ and $\beta$ respectively with the vertical, then $\tan\beta=3\tan\alpha$. Find the tension in the string.

**20.** A series of 5 equal uniform rods are jointed together, and one extremity of the series is fixed. Each rod, except the one with a fixed end, has a support placed under it, so that the rods rest in a horizontal position and the pressures on the supports are all equal. Show that the point of support of the 3rd rod from the fixed end divides its length in the ratio 2 : 7.

**Concurrent Forces in Three Dimensions.**

*Experiment.* Fasten three strings to the ring of a kilogram weight, and carry these strings over three well-mounted pulleys not situated in the same vertical plane. Fasten weights P, Q, R to their extremities, and adjust until the system is in equilibrium.

On a sheet of drawing paper placed on the table below, draw the projections of AB, AC, AD.

Find the angles $\alpha$, $\beta$, $\gamma$ which AB, AC, AD make with the vertical, and draw $OP_1 = P \sin \alpha$, $OQ_1 = Q \sin \beta$, $OR_1 = R \sin \gamma$.

Show, by drawing a triangle of forces, that $OP_1$, $OQ_1$, $OR_1$ represent a system in equilibrium.

Show also that $P \cos \alpha + Q \cos \beta + R \cos \gamma = W$.

If we take O as origin and draw two axes OX, OY at right angles to one another in the horizontal plane and OZ at

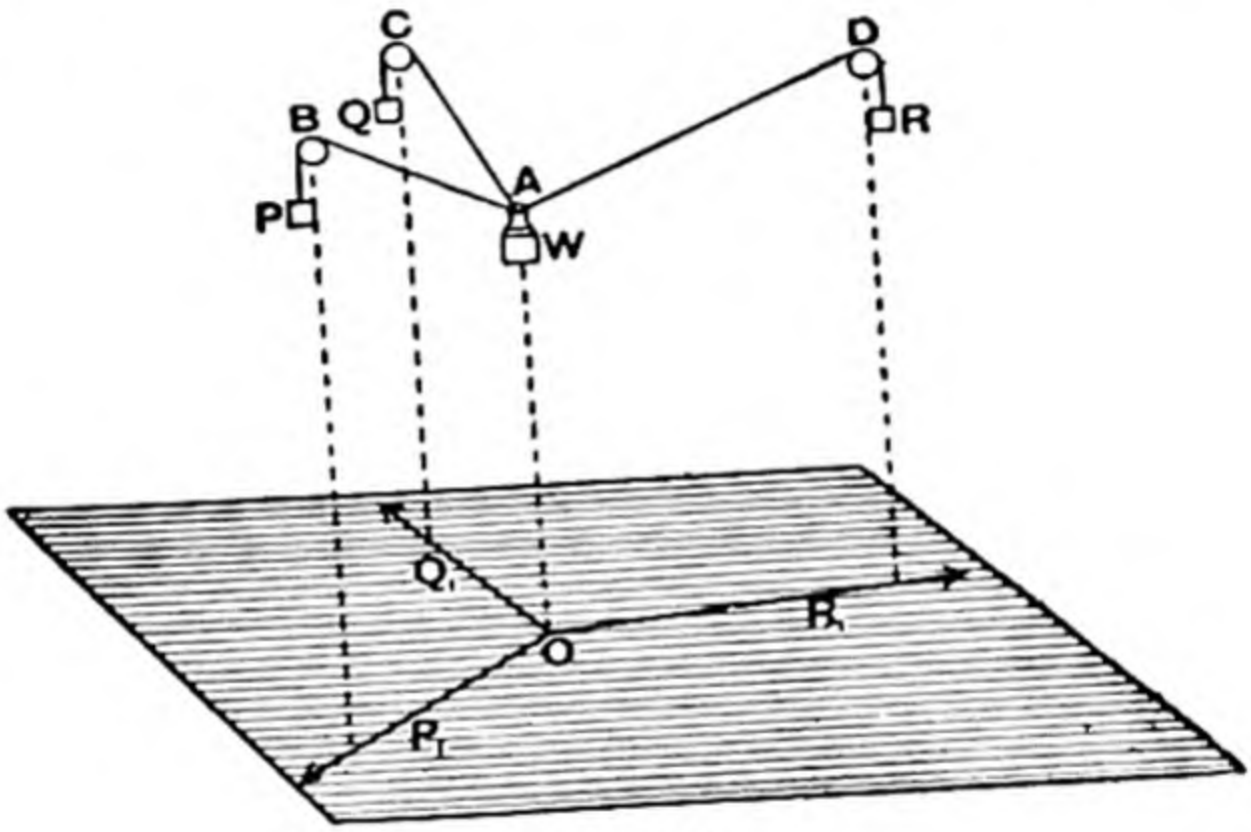



FIG. 240.

right angles to the plane, we have, if X and Y represent the resolved parts of $OR_1$, $OP_1$, $OQ_1$ along these axes and Z the vertical components of P, Q, R,

$$X = 0, \quad Y = 0 \text{ and } Z = W.$$

The following example illustrates the method of dealing with three concurrent forces when they can be reduced to a system of coplanar forces.

Three light poles of equal length, 6 ft., are fastened at O and placed with their other ends A, B, C on a horizontal rough

plane, so that ABC is an isosceles triangle. BC = AC = 5 ft., AB = 6 ft. If a weight of 20 lbs. is hung from O, find the thrusts in the poles.

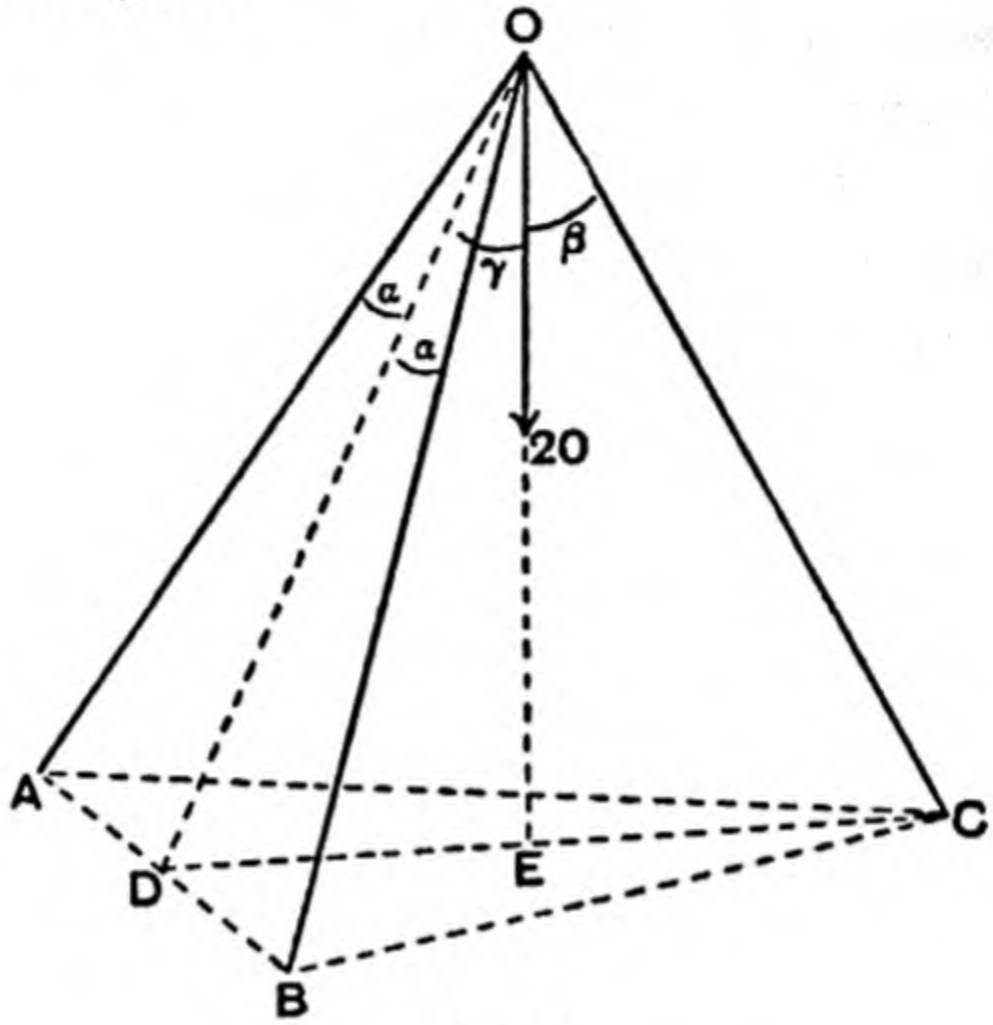



FIG. 241.

The resultant (R) of the thrusts along AO and BO will, by symmetry, bisect the angle AOB and lie in the same plane as W and OC.

*Solution by Calculation.*

Since OA = OB = OC, O will be vertically over the centre of the circumscribing circle of ABC.

Hence $EC = \frac{2.5}{8}$.

If P be the thrust in AO and in BO, and Q the thrust in CO, we have :

Resolving vertically,

$$Q \cos\beta + 2P \cos\alpha (\cos\gamma) = 20. \quad \text{.....(i)}$$

*N.B.*—If OA, OB are inclined at an angle $\delta$ to the vertical, we may write $Q \cos\beta + 2P \cos\delta = 20$ instead of (i).

Resolving horizontally in the plane DOC, we have

$$Q \sin\beta = 2P \cos\alpha \sin\gamma. \quad \text{.....(ii)}$$

By calculation, we find $OE = 5{\cdot}1$, $OD = 3\sqrt{3}$, $DC = 4$.

Equation (ii) gives $Q = \frac{14}{5}P$.

Hence $P = 9{\cdot}2$ lbs., $Q = 5{\cdot}2$ lbs.

*Graphical Solution.*

Find, by drawing, the altitudes OD and CD of the triangles AOB, ACB. Draw the triangle DOC. Construct the triangle of forces for the three forces, 20 lbs., Q and the resultant R of the thrusts in AO and BO. Hence find R. In the triangle AOB, cut off a length along the bisector OD to represent R, and hence, by a triangle of forces, find P, the thrust in AO and BO.

## EXAMPLES XXXI.

**1.** From three points A, B, C in a horizontal plane forming an equilateral triangle of side 4 ft., three strings of length 5 ft. are suspended and joined together at O, from which a weight of 5 lbs. is hung. Find the tension in the strings.

**2.** A tripod consisting of 3 struts, each 20 ft. long, is used to lift a body weighing 2 tons. The struts are equally inclined to the ground, and their ends lie on a circle of 5 ft. radius. Find the thrust in the struts.

**3.** Two spars AB, AC, each of length 20 ft., are fixed with C and B 10 ft. apart, on a horizontal plane, and A is tilted over until it is 15 ft. above the plane, being supported by a backstay AD fastened to D on the ground equidistant from B and C and 40 ft. from BC. If 2 tons is suspended from A, find the thrusts in the spars and the tension in AD.

**4.** A horizontal wire stretched to a tension of 500 lbs. is fastened to the top of a vertical pole 24 ft. high, which has two back guys, each 40 ft. long, fastened to the ground at points 20 ft. apart and symmetrically situated with respect to the horizontal wire. Find the tension in these guys if there is to be no horizontal pull on the pole.

**5.** A tripod is formed of three uniform rods of equal length smoothly jointed together at their top ends. It is set upon a smooth floor so that the feet are at the corners of an equilateral triangle, and the feet are joined by three chains of equal length. The rods are inclined at 30° to the vertical; the weight of each is 1 lb., and a weight of 5 lbs. is hung from the apex of the tripod. Find the tension in the chains. (C.U.)

**6.** Three equal bars are joined together to form an equilateral triangle of side 1 foot, which is held horizontal. A sphere weighing 2 lbs., of radius $1\frac{1}{2}$ feet, rests on the three bars. Find the normal pressure on each bar.

**7.** Three equal smooth spheres on a smooth horizontal plane are in contact with one another, and are kept together by an endless string in the plane of their centres, just fitting them: if a fourth equal sphere be placed on them, show that the tension of the string is to the weight of either sphere as $1 : 3\sqrt{6}$.

**8.** A V-shaped trough of angle 60°, with a vertical triangular end, has its side faces inclined at 60° to the horizontal plane. A smooth sphere weighing 1 lb. is placed in it, and the trough tilted until its parallel edges make an angle of 25° with the horizontal, so that the sphere rests against the triangular face. Find the pressure of the sphere on the side faces and on the end of the trough.

**Frameworks. Method of Sections.**

If it is required to find the force in any part of a framework without making a complete graphical solution, we may employ the method of sections.

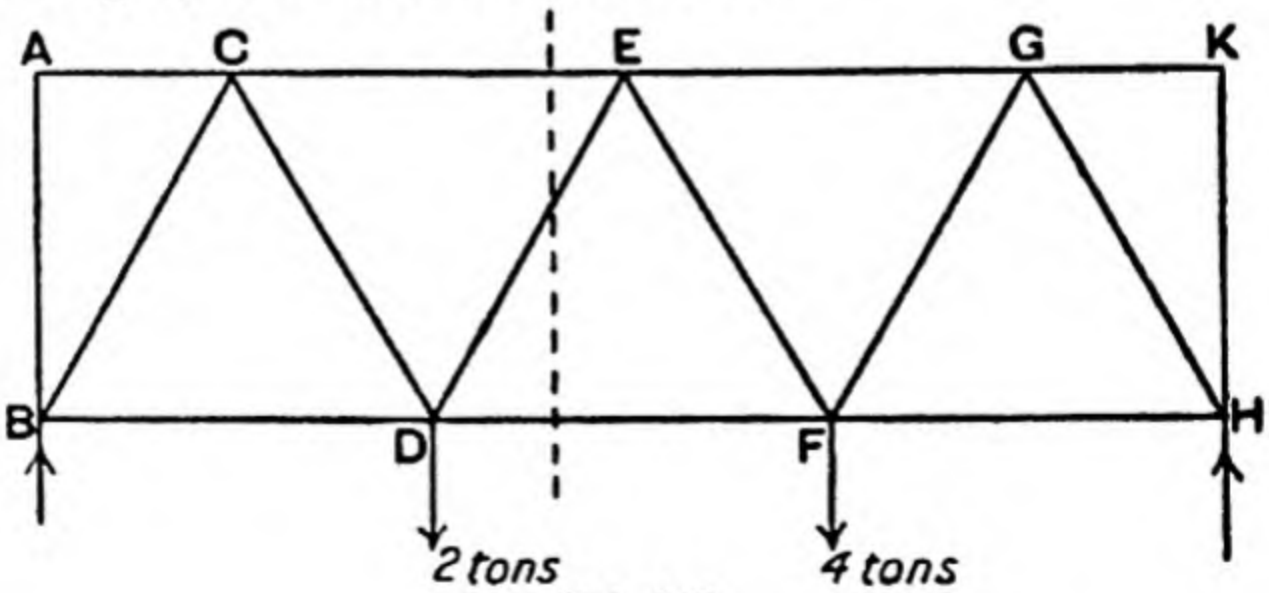



FIG. 242.

Suppose we require the force in the rod DE in the above framework.

If we can draw a line through the framework which does not cut more than three bars, including DE, and if these bars do not meet in a point, we may write down sufficient equations to determine the stresses in the bars.

Resolving horizontally, $P + R + \frac{Q}{2} = 0.$

Vertically, $\frac{8}{3} + Q \cdot \frac{\sqrt{3}}{2} = 2; \quad \therefore \quad Q = -\frac{4\sqrt{3}}{9}.$

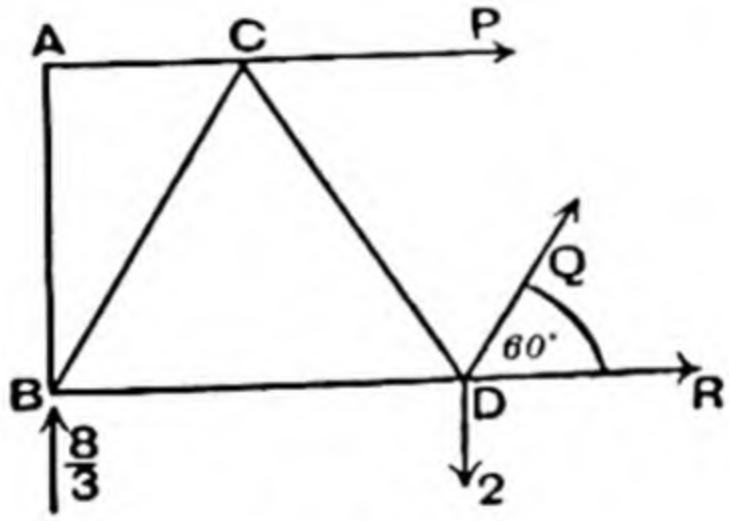



FIG. 243.

Taking moments about D,

$$P \cdot \sqrt{3} + \frac{8}{3} \cdot 2 = 0.$$

Hence $P = -\frac{16\sqrt{3}}{9}$ and $R = 2\sqrt{3}$;

P therefore acts towards C, and Q towards D, both being struts.

*N.B.* This method can only be adopted if the section cuts not more than three bars which do not meet in a point.

## EXAMPLES XXXII.

**1.** The framework consists of three equilateral triangles

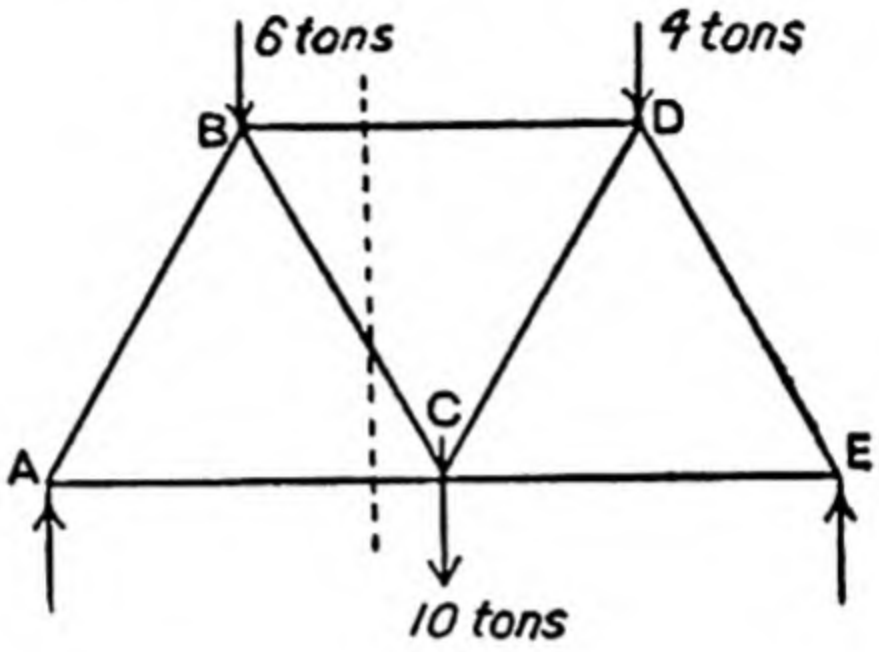



FIG. 244.

Find by the method of sections the forces in the bars BD, BC and AC.

**2.** The sloping bars in the N girder are inclined at 60° to the horizontal. Find the forces in the bars CD, CE, FE.

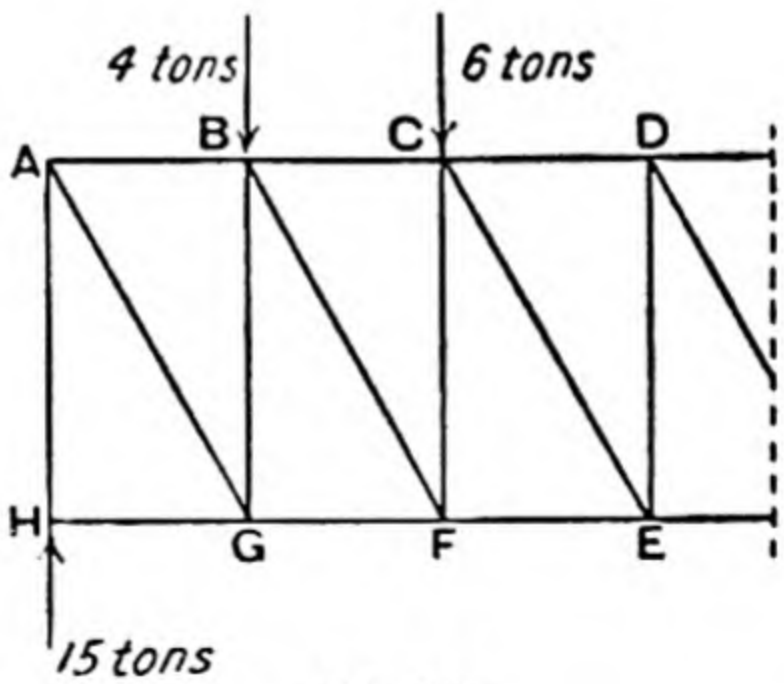



FIG. 245.

**3.** CAE = EAB = 15°, BE = AE. BE, BD are inclined at 45° to the horizontal.

Find the forces in AB, EB and ED.

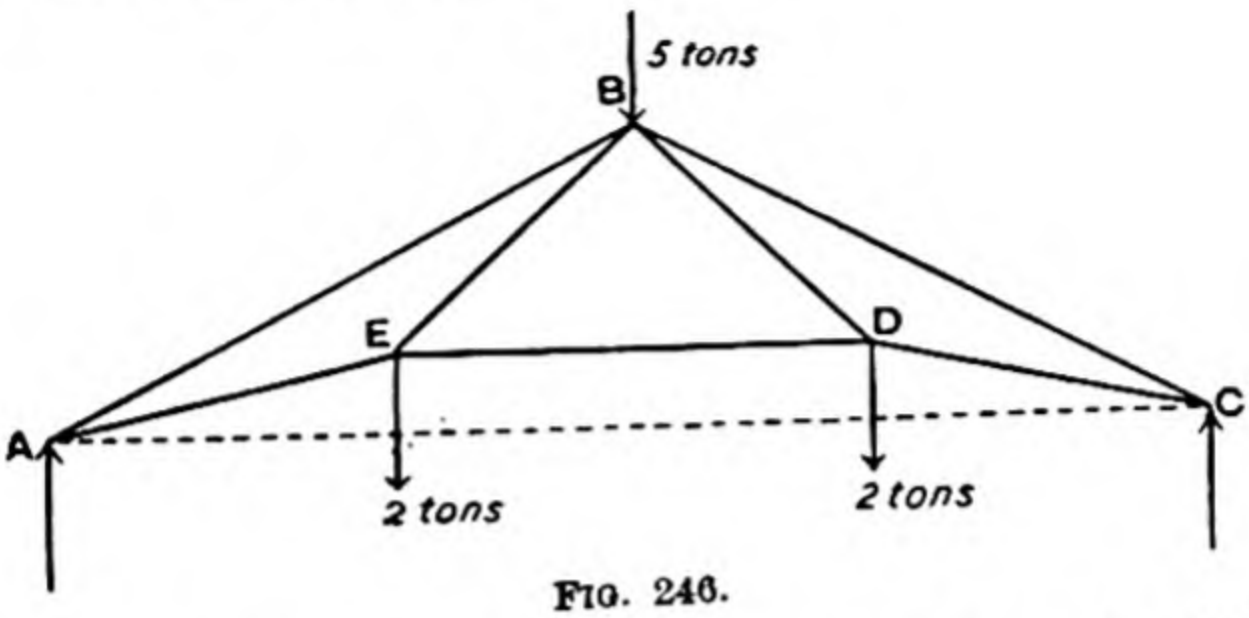



FIG. 246.

**4.** The bar CB is at right angles to AD at its mid point. Find the stress in the rods BE, BD, CD. (Fig. 247.)

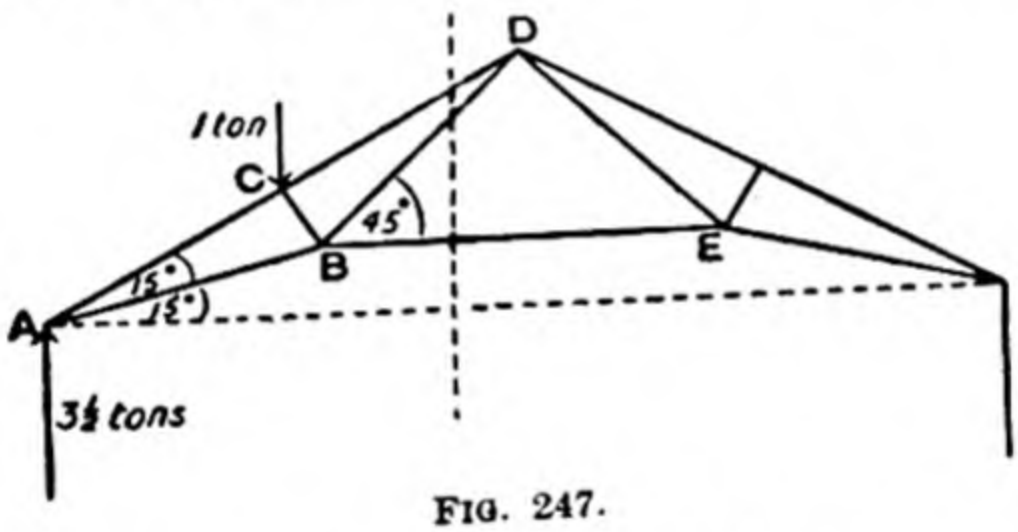



FIG. 247.

**5.** Find by the method of sections the stresses in AB, BC and CD when $\frac{1}{2}$ ton is placed at D. BC = BD = 3 ft., CD = 4 ft.

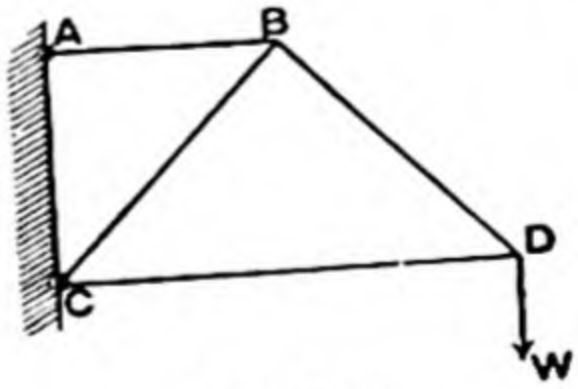



FIG. 248.

**6.** The horizontal roadway AG of a bridge is made up of 6 bars, AB, BC, etc., hinged together at their extremities,

each being 10 ft. long. From H and K on the vertical piers AH and GK, 20 ft. below the level of the bridge, struts HB, HC, HD, KD, KE, KF support the joints. If each joint carries a load of 1 ton and the stresses in AB, FG are taken as 2 tons each, find by the method of sections the stresses in HD and CD.

**General Conditions of Equilibrium.**

*Graphical Method.* To find the resultant of the three forces X, Y, Z acting on a rigid body.

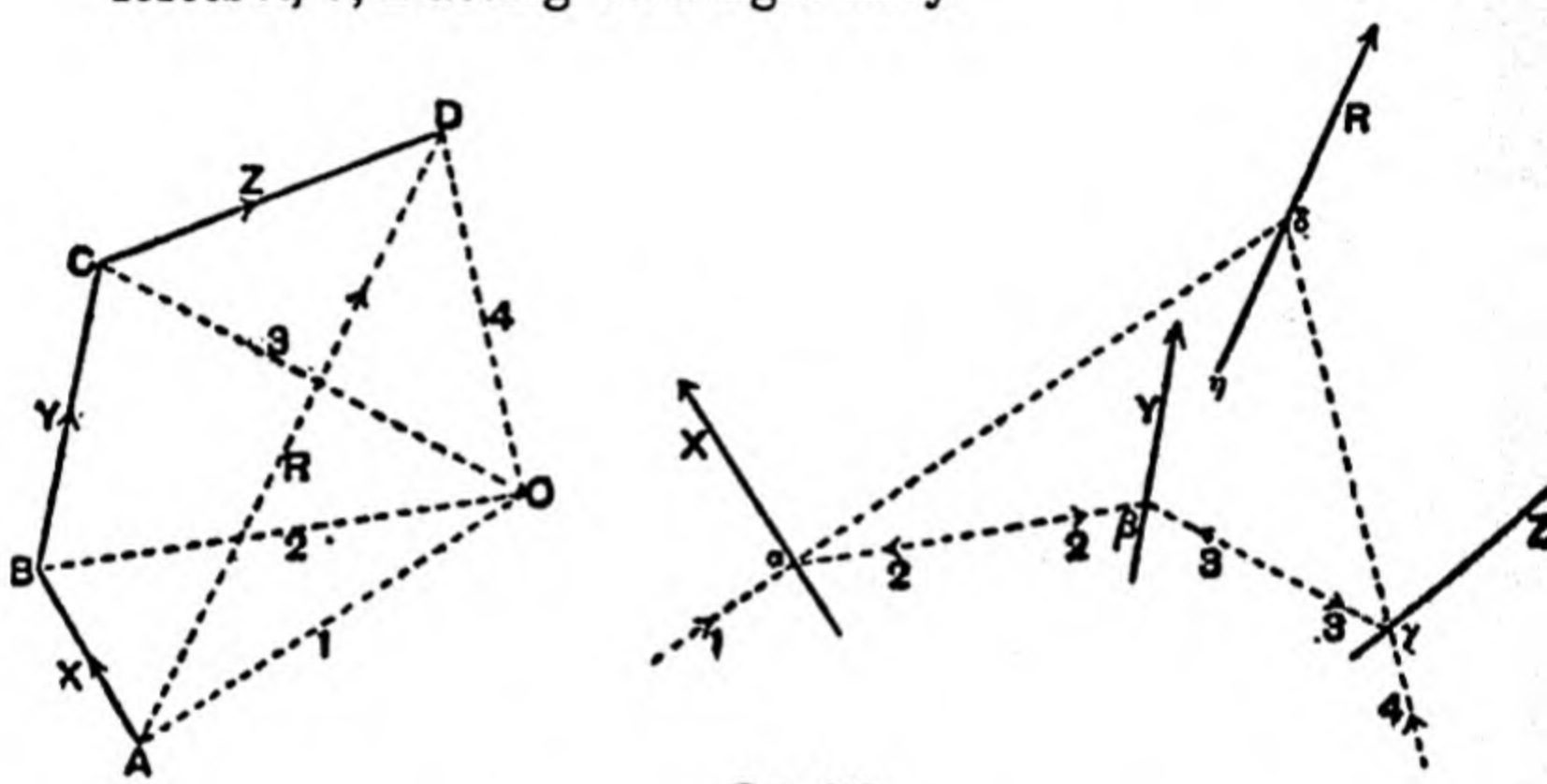



FIG. 249.

Draw a force diagram ABCD, representing the forces in magnitude and direction, and take any pole O.

Join OA, OB, OC, OD.

Through $\alpha$, any point on X, draw $\alpha\delta$ parallel to AO and $\alpha\beta$ parallel to BO, cutting Y at $\beta$.

Through $\beta$ on Y draw $\beta\gamma$ parallel to CO, and so on.

Join AD. Through $\delta$ draw $\delta\eta$ parallel to DA.

The given forces X, Y, Z are equivalent to

AO (1) along $\alpha\delta$, OB (2) along $\beta\alpha$,

BO (2) along $\alpha\beta$, OC (3) along $\gamma\beta$,

CO (3) along $\beta\gamma$, OD (4) along $\gamma\delta$,

*i.e.* to AO along $\alpha\delta$ and OD along $\gamma\delta$.

These, by the triangle of forces AOD, are equivalent to AD through $\delta$ parallel to AD, which is therefore the resultant of the given forces.

*Note* 1. If Z had been equal, parallel and opposite to the resultant of X and Y, we should have

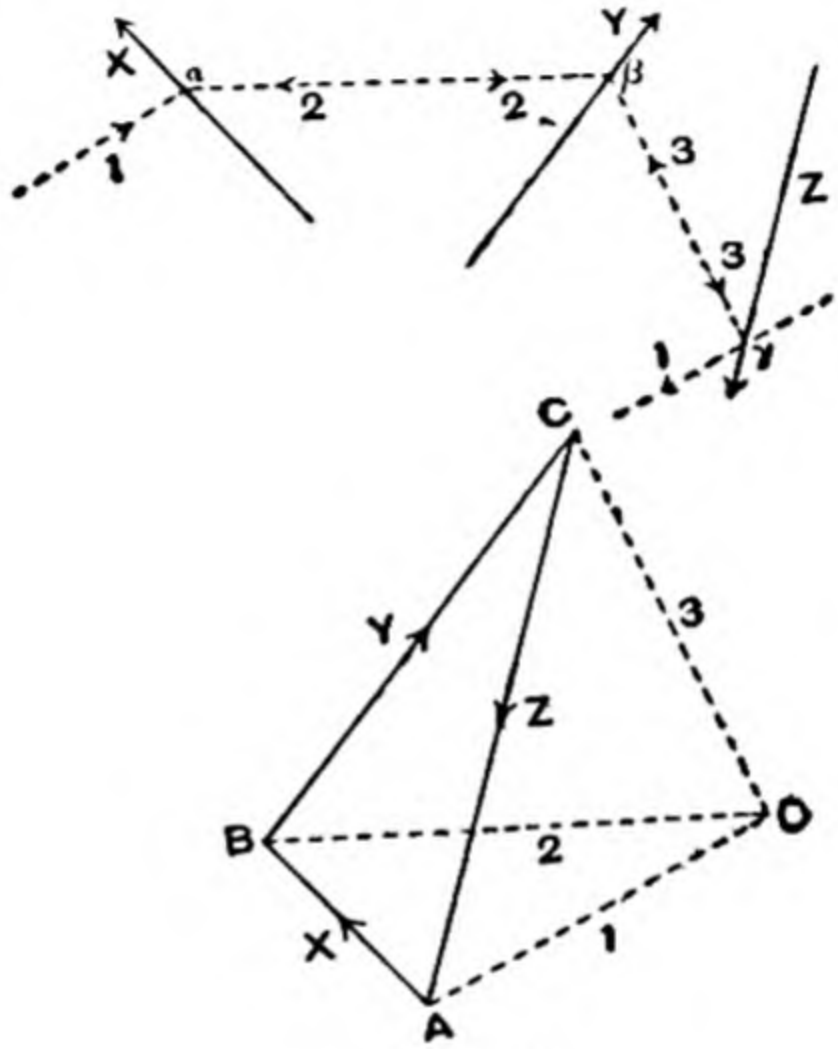



FIG. 250.

X at $\alpha$ is equivalent to AO (1) and OB (2).

Y at $\beta$ ,, BO (2) and OC (3).

Z at $\gamma$ ,, CO (3) and OA (1).

So that the system reduces to two unlike equal parallel forces AO at $\alpha$ and OA at $\gamma$.

In this case the link polygon is open but the force polygon closed, and the given system reduces to a couple.

*Note* 2. If, in addition to X, Y, Z in the original system, there had been a fourth force equal and opposite to the

resultant of X, Y, Z and acting along $\delta\eta$, both the force polygon and the link polygon would have been closed, and there would have been no resultant force.

## EXAMPLES XXXIII.

**1.** Draw a square ABCD of side 2″. A force of 3 lbs. acts from C to A, 5 lbs. along AB, 4 lbs. along BC and 4 lbs. along ED joining E, the mid point of BC, to D. Find graphically the line of action and magnitude of their resultant.

**2.** ABCDEF is a regular hexagon of side 2″. A force of 4 lbs. acts along FA, 6 lbs. along CB, 4 lbs. along BE and 5 lbs. along DF. Find graphically the magnitude and line of action of the resultant.

**3.** A number of strings are knotted at B, C and D, and pass over smooth pulleys at A, E and F. The weight hanging at

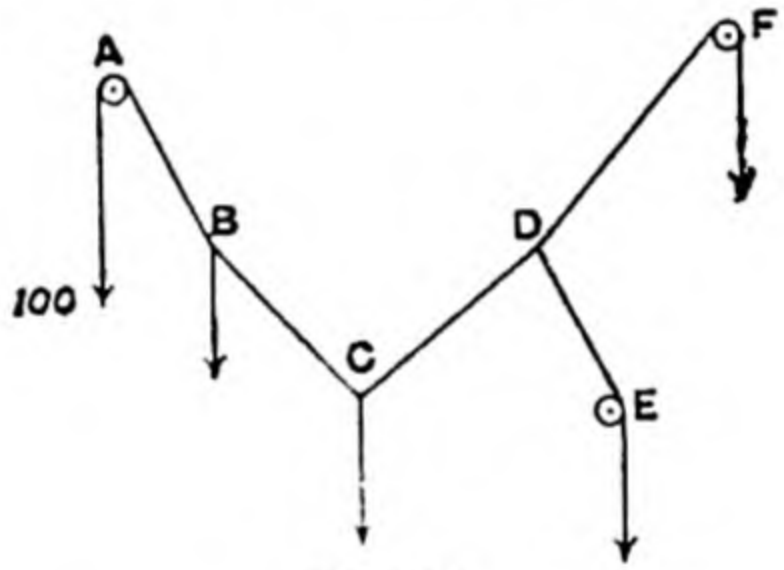



FIG. 251.

A is 100 grams. Find graphically the other weights, given that the inclinations of the lines to the vertical are AB 30°, BC 45°, CD 50°, DF 40°, and that CDE = 80°.

**4.** Three forces act on a rigid body along the sides of an equilateral triangle whose side is 2″, 4 lbs. along AB, 5 lbs. along BC and 3 lbs. along AC. Find graphically the magnitude and line of action of the resultant.

Solve also by calculation (see Ch. VI. p. 168).

**5.** Forces of 2 lbs., 3 lbs., 4 lbs. act respectively along the sides AB, BC, CD of a square. Find their resultant graphically, and hence reduce them to a single force through A and a

couple. State the moment of the couple if the side of the square equals 1 foot.

**6.** Along a line AE mark off $AB = BC = CD = DE = \frac{1}{2}''$. Suppose forces $P = 3$ lbs., $Q = 2$ lbs., $R = 2{\cdot}5$ lbs., $S = 2$ lbs., $T = 3{\cdot}5$

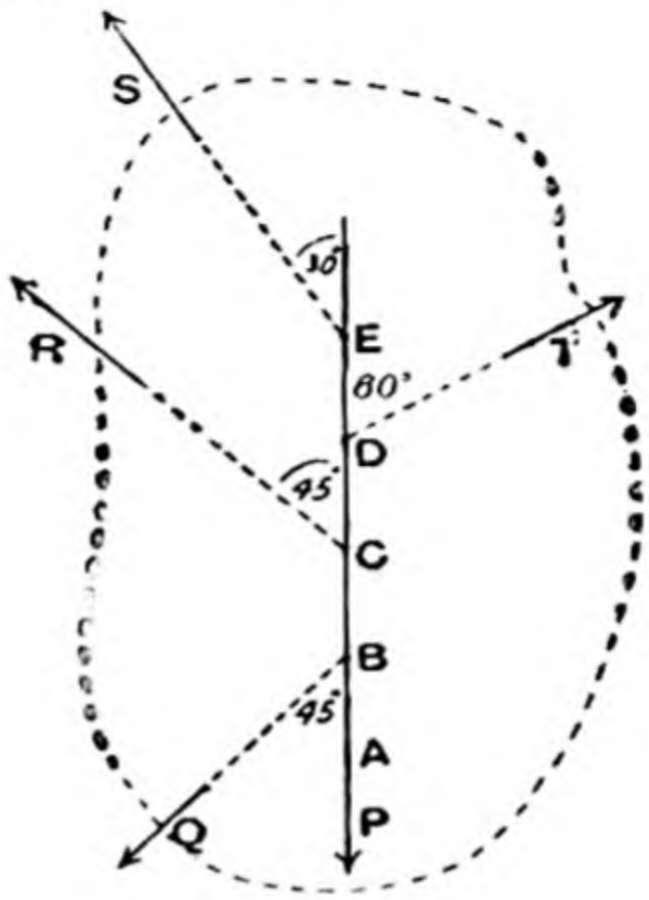



FIG. 252.

lbs. to act upon a rigid body as in the figure. Find graphically their resultant and the distance from A of the point where its line of action cuts AE.

**7.** If all the forces acting on a rigid body are known except three, and it is known that these three act along given lines (not concurrent), the magnitudes of these three may be found so that the system may be in equilibrium.

Draw a square ABCD of side 2″. Bisect AB at E, AD at G, DC at H. $P = 2$ lbs. acts along HB, $Q = 2$ lbs. along GC, R along AH, S along EG, and T makes 30° with DA at D, outside the square. Find the resultant of P and Q, and let its line of action cut R at O. If T and S meet at X, their resultant must act along OX.

Draw the triangle of forces for the resultant of P and Q, the force R and a force along OX. Hence find S and T.

**8.** Forces 1, 2, 3, 2 act along four consecutive sides AB, BC, CD, DE of a regular hexagon in order. Find graphically the magnitude and line of action of their resultant.

# CHAPTER X.

## FRICTION.

THE following examples illustrate methods adopted when dealing with more difficult questions involving friction.

EXAMPLE 1. AB represents a ladder resting against a rough wall at A and on a rough horizontal plane at B. To determine how far a man may ascend the ladder before it slips.

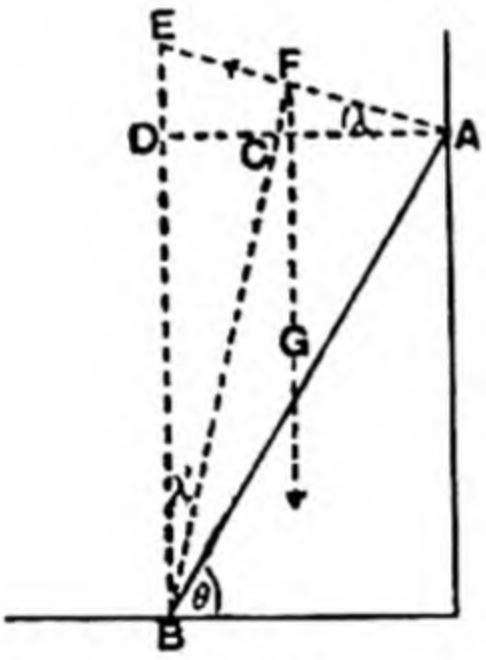



FIG. 253.

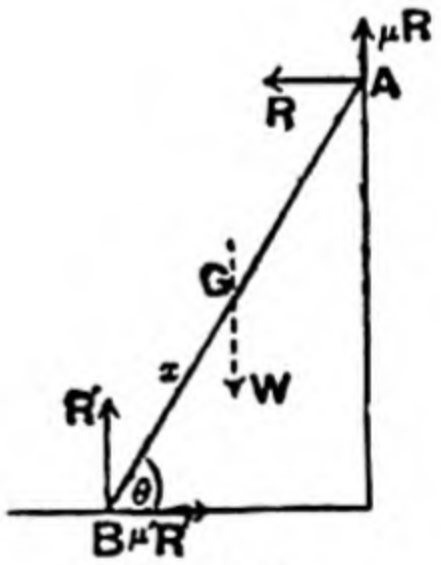



FIG. 254.

Let $\lambda$ be the angle of friction at A, and $\lambda'$ at B. The resultant action at A can act within the angle DAE, and at B within the angle DBC.

If the vertical through G, the C. of G. of man and ladder, cuts the quadrilateral EDCF, equilibrium may be obtained.

When the ladder is on the point of slipping, both at A and B, the vertical through G will pass through F. The man may therefore ascend until the C. of G. of man and ladder is vertically below F.

friction is in action at C and E, and for equilibrium the line of action of P must pass through O.

Then $$AM = \frac{AG + AB}{2} = \frac{\frac{x}{\mu} + a}{2} = \frac{x + a\mu}{2\mu}.$$

This requires that $\mu = \frac{x}{2AM - a} = \frac{x}{c}$, where $c$ is the distance between the two handles.

If the drawer does not jam at first when $x = b$, it is therefore likely to jam when the drawer has been pulled out some distance.

If $\mu < \frac{x}{c}$, the drawer will slip even though P acts at a distance from A greater than $x \cot \lambda$.

**Analytical Solution.**

Resolving parallel to AB, we have

$$R = R_1.$$

When the drawer is on the point of moving,

$$F = \mu R \text{ and } F_1 = \mu R_1;$$

$$\therefore F = F_1.$$

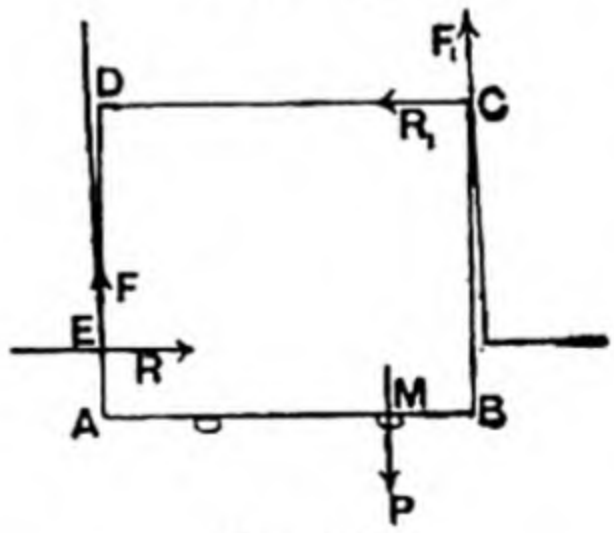



FIG. 256

The drawer will open if $P > F + F_1 > 2F$.

Take moments about E. Let DE be $x$.

$$Fa + Rx = P \cdot AM;$$

$$\therefore P = \frac{Fa + Rx}{AM}.$$

The drawer will therefore open if $\frac{Fa+Rx}{AM}>2F$,

*i.e.* if $$\mu Ra+Rx>2AM(\mu R)$$

or $$AM<\frac{x+a\mu}{2\mu},$$

whatever the value of $\mu$.

If AM is given, the condition for the drawer to jam is

$$\mu>\frac{x}{2AM-a},$$

$$>\frac{x}{c},$$

where $c$ is the distance between the handles.

EXAMPLE 3. Friction at the axle of a pulley working in a journal. The axle fits loosely in the journal, so that they may be considered as in contact at only one point A.

Let P be the tension in the string which is just raising a weight W. P is acting against friction, and is therefore greater than W.

Let N be the normal pressure of the axle, then friction $=\mu N$, and the resultant of these is R, which must be vertical and will act on the same side of the vertical through A, as P.

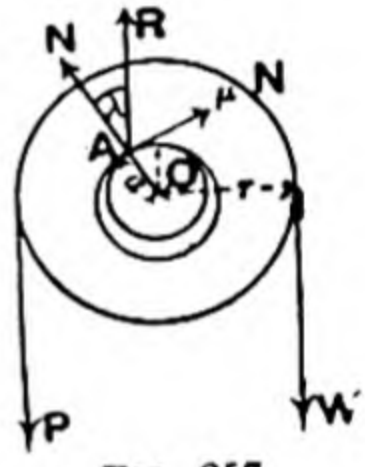


FIG. 257.

Resolving vertically, $R=P+W$. (The weight of the pulley is neglected.)

Taking moments about O,

$Pr=Ra\sin\lambda+Wr$, where $\lambda$ is the angle of friction;

$$\therefore P=\frac{W(r+a\sin\lambda)}{r-a\sin\lambda},$$

with its axis vertical.

EXAMPLE 4. A cylinder of radius $r$ is held between two rough vertical boards hinged to one another at an angle. Prove that the cylinder cannot be squeezed out unless its axis is at a distance from the line of hinges less than $r$ cosec $\lambda$.

The total reaction at each point of contact being R, these forces can balance if they act along the same line, no matter how great they may be. When the cylinder is on the point of slipping, each resultant reaction makes an angle $\lambda$ with the

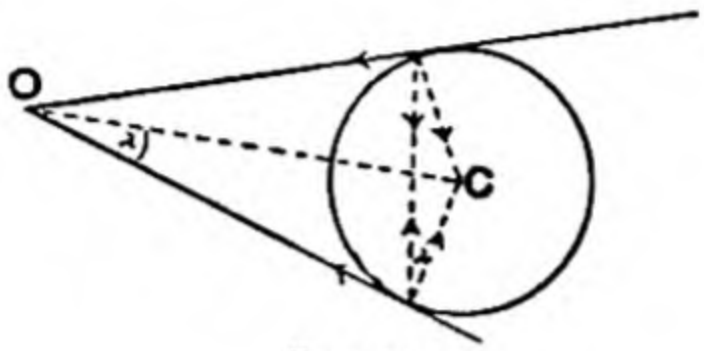



FIG. 258.

normal. The distance of C from O is then $r$ cosec $\lambda$. If the cylinder is nearer to O than this, the resultant forces will require to act at an angle to the normal greater than $\lambda$, in order to be in equilibrium, which will be impossible.

The investigation above is illustrated by the action of a bicycle when free-wheeling. The chain drives the inner toothed wheel, while the outer ring is connected to the driving wheel.

When the bicycle is driven to the left, the ball bearings jam,

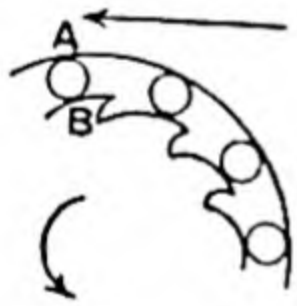



FIG. 259 (*a*).

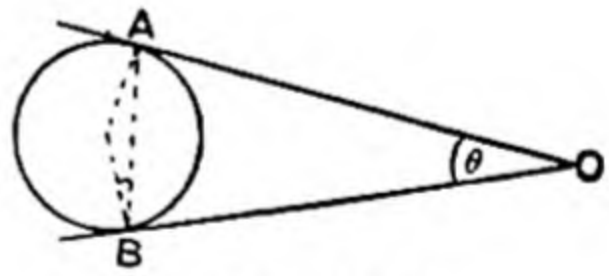



FIG. 259 (*b*).

and the drive on the chain wheel makes the outer wheel rotate to the left.

When free-wheeling the balls are carried into the wider spaces between the teeth, and the outer ring continues to rotate while the inner wheel is at rest.

If OA, OB represent the tangents to the roller, where it is in contact with the outer ring at A and the teeth at B, the roller will jam if $\theta$ is not greater than $2\lambda$.

It is important to notice that the resultant of the normal pressure and the friction will balance all other forces acting

on a body, so long as the resultant of these other forces acts within the angle of friction. The normal pressure may increase indefinitely (up to breaking point), and therefore the resultant of the normal pressure and maximum friction can increase indefinitely; but its direction is limited, since it cannot make an angle with the normal greater than $\tan^{-1}\mu$.

EXAMPLE 5. Two equal rough cylinders of weight W are placed upon a horizontal plane with their axes parallel and a plank of weight W′ is laid across them perpendicular to their axes, so as to project equally beyond each. The plane is then tilted about a line parallel to the axes, while the upper cylinder is held fixed. Prove that if $\mu$ is the same at all points of contact, equilibrium will be broken when the inclination of the plane to the horizontal is $\tan^{-1}\dfrac{\mu W'}{2W'+W}$.

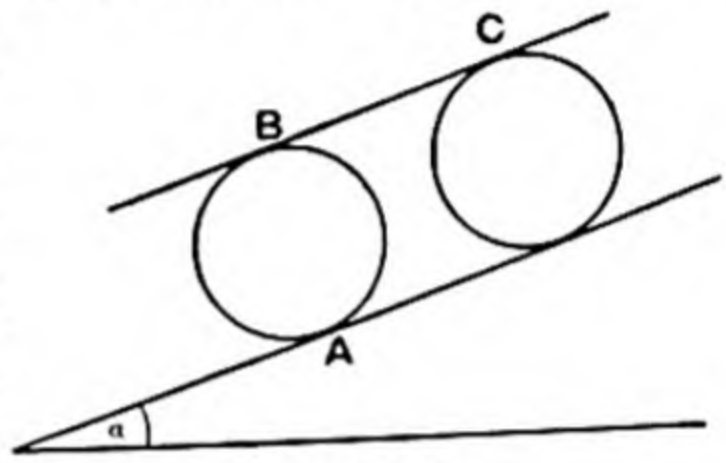



FIG. 260.

Consider the lower cylinder.

The friction forces at A and B will both act so as to prevent motion at these points if possible, *i.e.* they will act to the right and will intersect vertically above the centre at H. Hence

$\hat{ABH} > \hat{BAH}$.

∴ the cylinder is more likely to slip at B than at A.

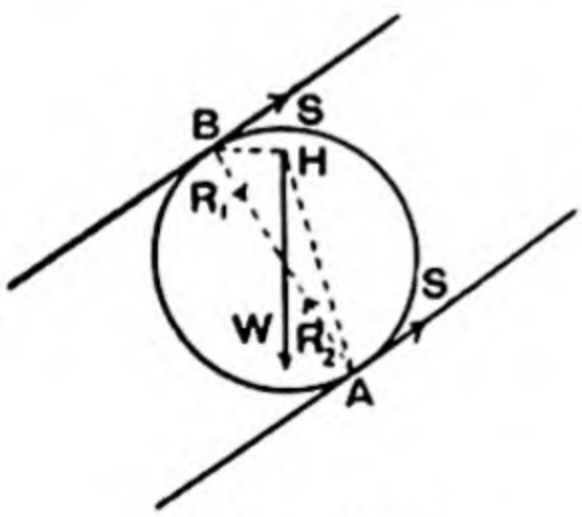



FIG. 261 (*a*).

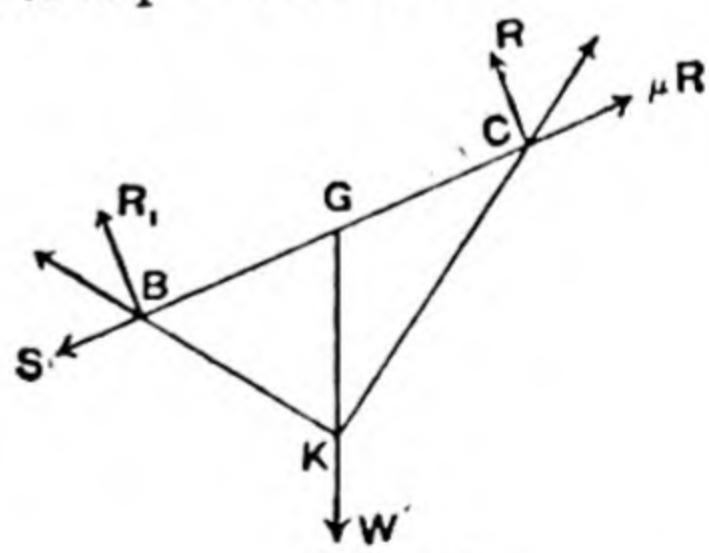



FIG. 261 (*b*).

Consider the beam. It is clear from the diagram that GCK < GBK. Therefore C is more likely to slip than B.

Hence C will slip before B and A; and when equilibrium is limiting, the friction at C will be $\mu R$.

$$\therefore \mu R = W' \sin\alpha + S \text{ for the plank,}$$

$$R_1 + R = W' \cos\alpha,$$

$$R = R_1 \text{ (taking moments about G).}$$

For the lower cylinder, $S_1 + S = W \sin\alpha$,

$S = S_1$ (taking moments about the centre).

Hence $$\frac{W \sin\alpha}{2} + W' \sin\alpha = \mu \frac{W' \cos\alpha}{2},$$

*i.e.* $$\tan\alpha = \frac{\mu W'}{2W' + W}.$$

## EXAMPLES XXXIV.

**1.** A right circular cone, vertical angle $2\alpha$, rests with its base on a rough horizontal plane; a string is attached to its vertex and pulled in a horizontal direction with a gradually increasing force; prove that, if $\alpha$ be $< \lambda$, the cone will tilt, and if $\alpha > \lambda$ it will slide.

**2.** A sphere rests on a rough horizontal plane, and its highest point is joined to a peg fixed in the plane by a tight cord parallel to the plane. The plane is tilted about its lower edge, which is perpendicular to the cord; find the inclination of the plane when the sphere slips, if $\mu = 0{\cdot}4$.

**3.** A rectangular box stands 8 in. from a smooth wall and parallel to it. Its lid, which is open and rests against the wall, weighs 40 lbs. If the box is 1 ft. from front to back and weighs 100 lbs. without the lid, find the least coefficient of friction between the box and the floor necessary to prevent motion.

**4.** Two rings, each weighing 4 oz., are connected by a string passing over a smooth peg. A uniform bar, weighing 10 lbs., is placed through the rings and supported horizontally. Prove that the least angle the string can make with the horizontal is $\tan^{-1} 2{\cdot}1$, if the coefficient of friction between the rod and the rings is $0{\cdot}5$.

**5.** A boy of weight W, standing on a sheet of ice, balances himself by means of a chair of weight W′, but does not lean any of his weight upon it. Assuming the weight of the boy to act halfway between his feet and his arms, prove that he may incline his body to the vertical at angle $\tan^{-1} 2\mu$, if W′>W, but only at an angle $\tan^{-1}\frac{2\mu W'}{W}$, if W>W′. $\mu$ is the coefficient of friction between the boy and ice and between chair and ice.

**6.** A bracket is supported by two rings 6 in. apart, which slide on a vertical rod. If the diameter of the rod is $\frac{1}{2}$ in., prove that the bracket will slide down unless the vertical through its C. of G. is more than $5\frac{3}{4}$ in. from the nearer edge of the rod. ($\mu = \cdot 5$.)

**7.** A reel of cotton has ends of radius 1 in., and the cotton is wound round a drum of radius $\frac{3}{4}$ in. If the reel is on a horizontal plane and is pulled along by the cotton, which leaves the reel at $\theta$ to the horizontal, prove that the reel will roll forwards or backwards, according as $\cos\theta \gtrless \frac{3}{4}$.

**8.** A weight can be supported on a rough inclined plane by a force P acting along the plane, or by a force Q acting horizontally. Show that the weight is $\frac{PQ}{\sqrt{Q^2\sec^2\phi - P^2}}$, where $\phi$ is the angle of friction.

**9.** Two equally rough pegs A and B are fixed in a smooth wall, B being higher than A, so that AB is inclined at 60° to the horizontal. A long heavy stick is placed so that it rests on B and presses against the lower side of A. The C. of G. of the stick is at a distance from B equal to AB. Find the least value of $\mu$, if the stick does not slip.

**10.** A square board ABCD rests in a vertical plane at right angles to a rough wall, with which A is in contact. It is supported by a rough peg beneath BC, close to C. If the coefficients of friction at the wall and at the peg are the same, find its least value for equilibrium to be possible with BC horizontal.

**11.** When a bicycle is on the level, the vertical through the C. of G. of the bicycle and rider, which is $h$ ft. from the ground, is $b$ ft. from the centre of the front wheel and $a$ ft. from the

back wheel. If $\mu$, the coefficient of friction between the ground and the back wheel, is $< \frac{a}{h}$, prove that the greatest slope the bicycle can climb is $\tan^{-1} \frac{\mu b}{a + b - \mu h}$.

If $\mu > \frac{a}{h}$, find the steepest hill that can be climbed. Can the bicycle climb a steeper hill if the rider leans forward over the handle-bars ?

**12.** A window, 3 ft. broad and 4 ft. high, of weight W, is supported by two vertical cords passing over smooth pulleys and fastened to two weights, each equal to $\frac{W}{2}$. If one cord breaks, find the least coefficient of friction between the sash and the window, in order that the window may not slide down

**13.** A ladder 20 ft. long, with its C. of G. 8 ft. up, weighs 60 lbs., and rests at an angle $\theta$ to the ground, against a smooth vertical wall. The coefficient of friction between the ladder and the ground is $\frac{1}{2}$. Find the least value of $\theta$ which will enable a man weighing 140 lbs. to reach the top without the ladder slipping.

**14.** A heavy block is supported on two rough wedges, the wedges resting on a rough horizontal table. $\mu$ is the same for

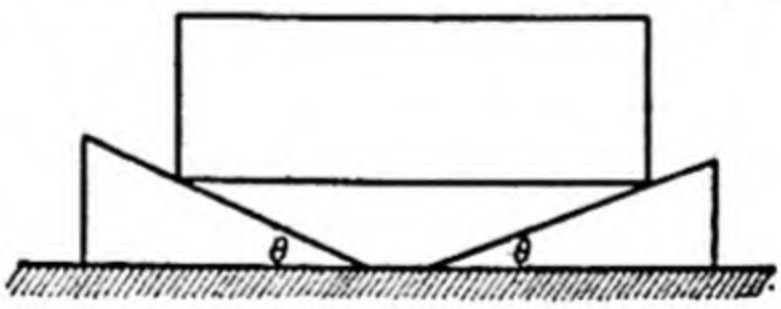



FIG. 262.

block and wedge as for wedges and table. Show that the block cannot cause the wedges to slip unless the angle $\theta$ is greater than twice the angle of friction. (C.U.)

**15.** A heavy uniform beam AB, 10 ft. long, rests on rough ground, and a force P acts parallel to the ground at right angles to the beam at B. Assuming that friction acts equally along the length of the beam, and that when P is gradually increased, the beam will turn round some point in its length, prove that the distance of this point from A is 2·93 ft.

**16.** The maximum slope on which a four-wheeled truck can stand when all wheels are locked is $\alpha$. When only the upper pair is locked, the maximum slope is $\beta$, and in this case the vertical through the C. of G. cuts the line joining the axles in the ratio 1 : 2. Prove that $\tan\beta = \frac{1}{3}\tan\alpha$, if the wheels are small.

**17.** A plane is inclined to the horizontal at an angle of 40°. A uniform chain of length 4 ft. lies along a line of greatest slope and hangs over the top edge. Find the greatest and least lengths of the vertical portion of the chain, if the angle of friction between chain and plane is 20°.

**18.** A uniform ladder 30 ft. long is being lowered from a vertical position by two men, one pushing in a direction perpendicular to the length of the ladder at a point on the ladder vertically distant from the ground his own height 6 ft., and the other exerting a horizontal force at the foot of the ladder. If $\mu = 0{\cdot}2$ between the ladder and the ground, show that the man at the foot must exert a force $\cdot 85W$ when the ladder is inclined at 60° to the horizontal, where W is the weight of the ladder. (C.U.)

**19.** A smooth circular cylinder 120 cms. in diameter lies on the ground and is held fast. A uniform plank 3 m. long, weighing 100 Kg., is placed against the cylinder and rests with one end on the ground at a distance of 240 cms. from the lowest line of the cylinder. If the plank in this position is just on the point of slipping down against the friction of the ground, find $\mu$, and find what horizontal force must be applied to the lower end of the plank to make it just begin to slip up on the cylinder. (C.S.C.)

**20.** Show that a sash window of height $h$, counterbalanced by weights, cannot be raised or lowered by a vertical force, unless it is applied at a point whose distance from the middle is $< \frac{h}{2\mu}$, where $\mu$ is the coefficient of friction between the movable window and the frame. If a cord breaks, show that the window will fall unless its width is $> \frac{h}{\mu}$.

**21.** A small ring, weight W, is strung on a rough circular hoop of weight $\frac{W}{4}$, radius $a$, and the angle of friction is $\lambda$. The

hoop is hung on a rough peg. Show that the hoop will be in equilibrium with its centre at any distance $\not> \frac{4}{5} a \sin\lambda$ from the vertical through the peg, if the coefficient of friction between hoop and peg be $\not< \dfrac{4\sin\lambda}{\sqrt{16\cos^2\lambda+9}}$.

**22.** Two uniform rods AB, AC, of weights W and W′, are rigidly connected at A, and AB is placed over a peg. Show that there is in general a length $\dfrac{W'}{W+W'}\, p\mu$ on AB, such that if the peg touch AB anywhere within this space, the system will be in equilibrium, where $p$ is the perpendicular on AB from C and $\mu$ is the coefficient of friction.

**23.** The figure represents a horizontal block of wood ABCD, 6 in. by 18 in., which slides in a groove in a vertical block PQRS, which is also 6 in. broad. Suppose that the block ABCD

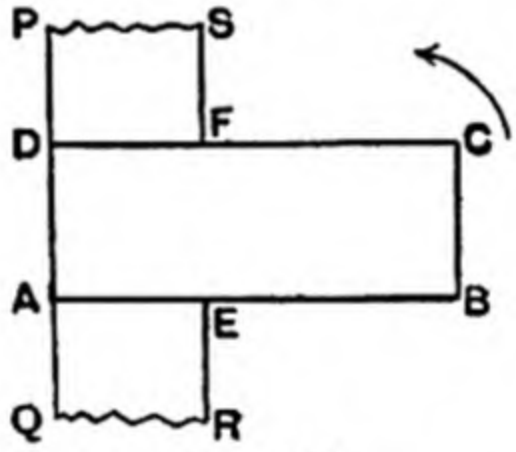



FIG. 263.

is in contact with PQRS only at the points D and E. Find the greatest angle through which the block PQRS can be tilted in the direction of the arrow without causing the block ABCD to slide. ($\mu = 0{\cdot}25$.) (C.U.)

**24.** A uniform beam AB of length $2a$ lies horizontally on two others at right angles to it, touching one at A and the other at C, $b$ ft. from A. A horizontal force P is applied at B and the beam turns round either A or C. Prove that the least value of P is the less of the two forces $\dfrac{\mu W(b-a)}{2a-b}$ and $\dfrac{\mu W}{2}$.

**25.** A uniform rectangular board ABCD rests with its corner A against a rough vertical wall and its side BC on a smooth peg, the plane of the board being vertical and perpendicular

to the wall. Show that the peg may be moved through a length $\frac{1+2\sqrt{3}}{8}$ without destroying equilibrium, where BC is inclined at 60° to the vertical. (AB = 1 ft., BC = 2 ft., $\mu = \cdot 5$.)

**26.** When an axle of radius $a$ revolves on bearings with which it has a coefficient of friction equal to tan $\lambda$, show that the resultant reaction acts in a line which is a tangent to a concentric circle of radius $a \sin \lambda$.

The bell crank lever shown in the figure is mounted on an axle of 3 in. diameter, and is subject to a vertical force Q, 5 in. from the centre, and a horizontal force P, 6 in. from the centre. If $\mu = \tan 30°$, find the value of P when just able to overcome a force Q = 10. (Fig. 264.) (C.U.)

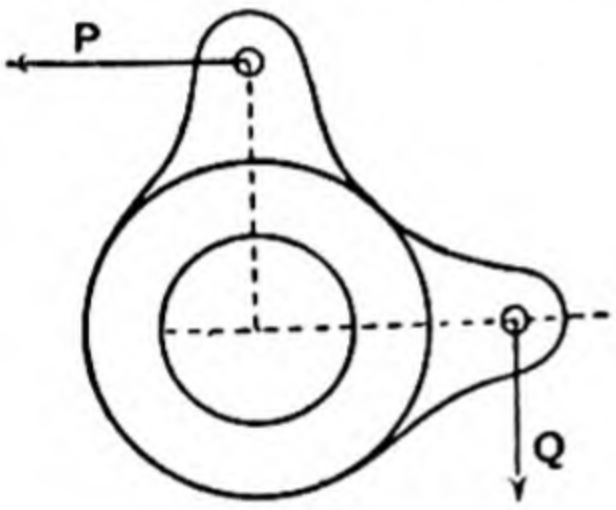



FIG. 264.

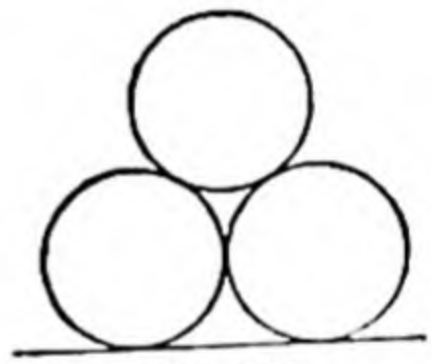


FIG. 265.

**27.** Three equal rough cylinders are in contact with one another and with a rough horizontal plane, as in the figure, all surfaces in contact being supposed equally rough. Show that, if the coefficient of friction be equal to $(2-\sqrt{3})$, the upper cylinder will be on the point of slipping between the lower cylinders, which will be on the point of rolling on the horizontal plane. (Fig. 265.)

**28.** Two circular cylinders of unequal radii rest on a rough horizontal plane with their axes parallel, and on them rests a rough uniform beam of weight W, being supported by a force parallel to its length. The axis of the beam is perpendicular to the axes of the cylinders and inclined at $a$ to the horizontal plane; prove

$$P = W \tan \frac{a}{2}.$$

# CHAPTER XI.

## CENTRE OF GRAVITY.

**Graphical Method of determining the C. of G.**

The method of finding the line of action of the resultant of a system of parallel forces given in Chapter VI. enables us to determine graphically the position of the C. of G. of a body.

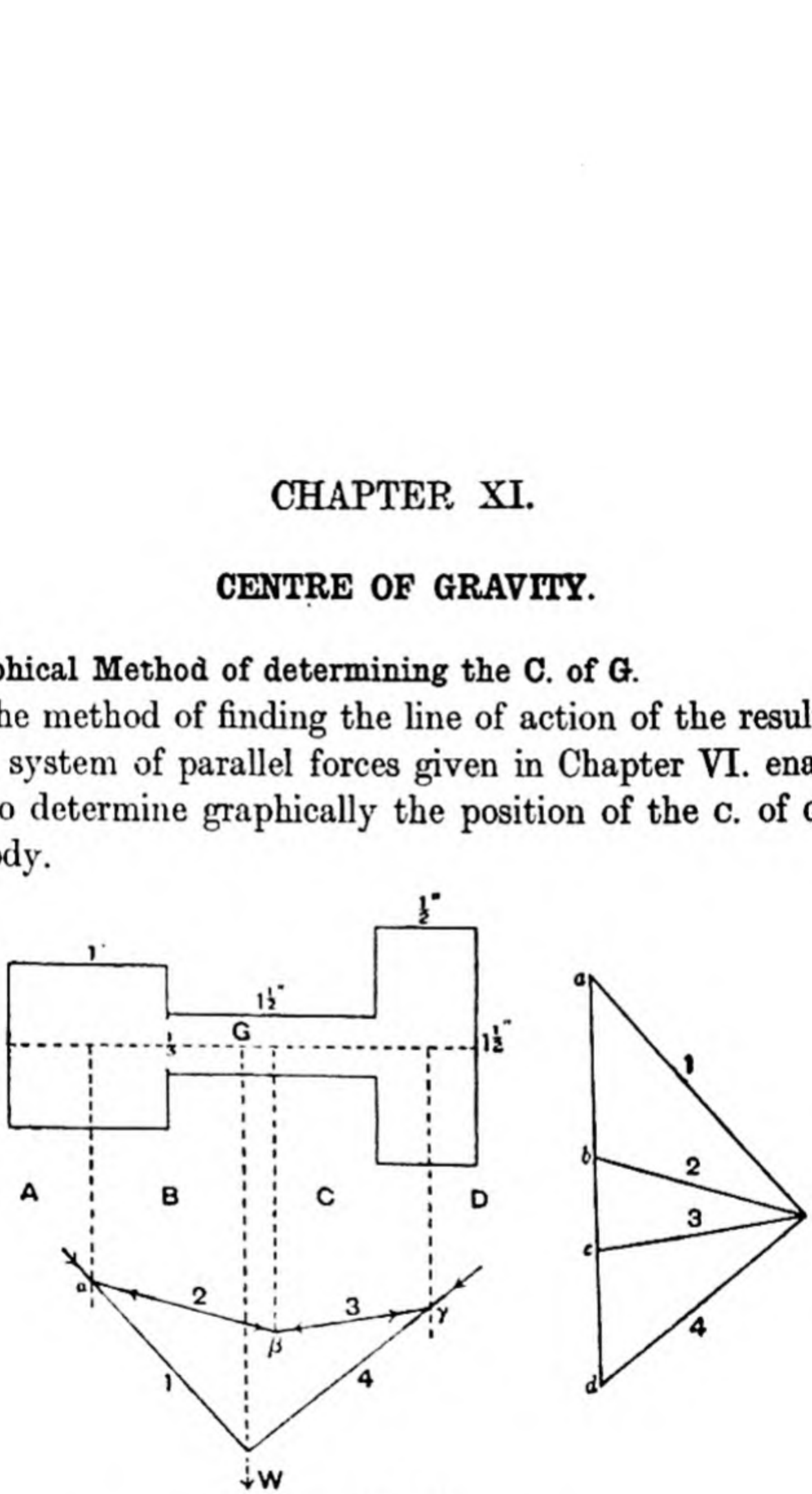



FIG. 266.

Draw the force polygon *abcd* for the weights of the three parts of the figure, then the link polygon $\alpha$, $\beta$, $\gamma$, and hence

determine the point through which the resultant of the three parallel forces will pass.

When the figure is of irregular shape, a line is drawn across it at right angles to two parallel tangents at the extreme edges. This line is subdivided into a number of equal parts (say six).

At the mid points of these subdivisions, lines are drawn across the figure, parallel to the tangents. The weights of

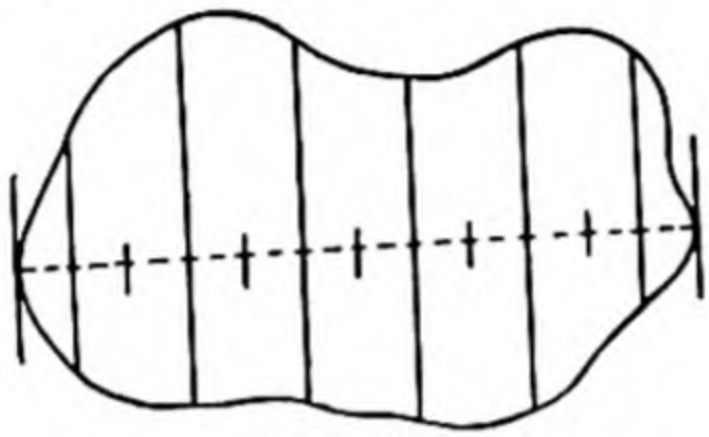


FIG. 267.

the six portions are taken to be proportional to the lengths of the lines so drawn, and a force polygon is constructed, as before, to determine the line of action of the resultant. The figure is again subdivided in another direction, generally at right angles to the former, and the line of action of the resultant of the parallel forces again determined.

The point of intersection of the two resultants will be approximately the C. of G. of the figure.

The coordinates of the C. of G. of an irregular figure may be obtained by calculation in the following way.

The figure, when drawn on squared paper, is divided into strips by the lines parallel to the Y-axis.

These strips are of equal width, except at the sides. Each strip of the figure, such as BCED, is considered as being of a uniform length, equal to the length of FK, midway between BC and DE, and the C. of G. of this strip is taken at $G_1$, where $FG_1 = G_1K$.

Since $\bar{x}=\dfrac{\Sigma wx}{\Sigma w}$ and $w_1=(1{\cdot}4)({\cdot}6)$, $x_1=({\cdot}7)$,

$w_2=(\mathrm{FK})(1)=2{\cdot}6$, $x_2=1{\cdot}5$, etc.,

$$\bar{x}=\frac{(1{\cdot}4)({\cdot}6)({\cdot}7)+(2{\cdot}6)(1{\cdot}5)+(3{\cdot}9)(2{\cdot}5)+(4{\cdot}2)(3{\cdot}5)+(2{\cdot}9)(4{\cdot}5)+(1{\cdot}5)({\cdot}8)(5{\cdot}4)}{(1{\cdot}4)({\cdot}6)+(2{\cdot}6+3{\cdot}9+4{\cdot}2+2{\cdot}9)+(1{\cdot}5)({\cdot}8)}$$

$$=\frac{48{\cdot}47}{15{\cdot}6}=3{\cdot}1.$$

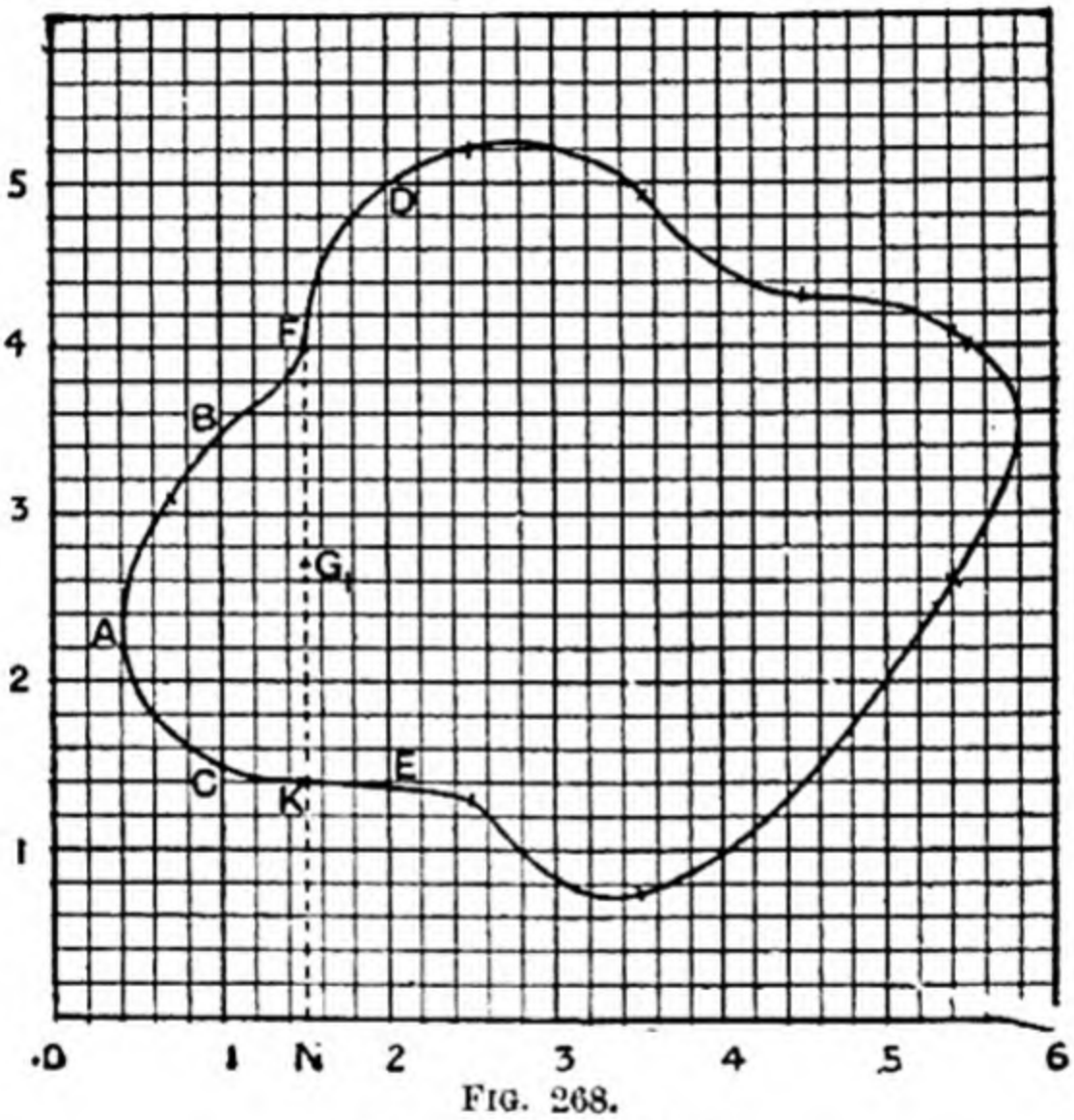



FIG. 268.

Also, $y_2=\mathrm{G_1N}=\dfrac{\mathrm{FN}+\mathrm{KN}}{2}=\dfrac{4+(1{\cdot}4)}{2}=2{\cdot}7$, etc.;

$$\therefore\ \bar{y}=\frac{(1{\cdot}4)({\cdot}6)(2{\cdot}4)+(2{\cdot}6)(2{\cdot}7)+(3{\cdot}9)(3{\cdot}25)+(4{\cdot}2)(2{\cdot}8)+(2{\cdot}9)(2\,85)+(1{\cdot}5)({\cdot}8)(3{\cdot}35)}{(1{\cdot}4)({\cdot}6)+(2{\cdot}6+3{\cdot}9+4{\cdot}2+2{\cdot}9)+(1{\cdot}5)({\cdot}8)}$$

$$=\frac{45{\cdot}76}{15{\cdot}64}=2{\cdot}93.$$

$\bar{y}$ could also be found by considering the figure to be divided into strips by the lines parallel to the X-axis.

If each strip has the same width, the areas may be considered as proportional to the lengths of the lines FK, etc., and the width of the strips will not appear. The narrower the strips, the more nearly correct will be the result.

The point thus obtained for a geometrical figure is more correctly termed *Centre of Area* or *Centroid*.

**To find the C. of G. of a Wire.**

Divide the wire AB into a number of elements of convenient length, say 1 unit of the squared paper. The part EB is here of length 0·4.

The C. of G. of each element is considered to be at its mid point.

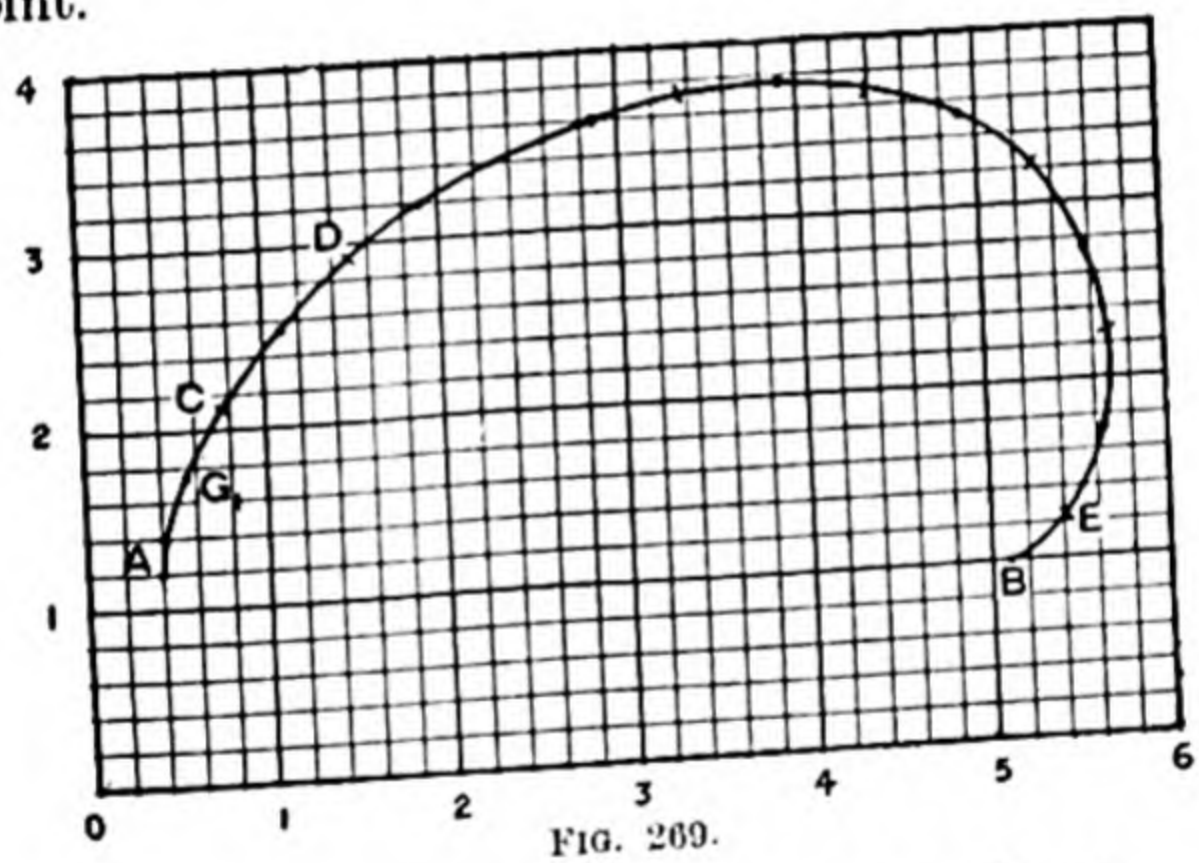



FIG. 269.

From readings taken from the figure, we have

$$\bar{x} = \frac{1(\cdot5 + 1\cdot1 + 1\cdot9 + 2\cdot85 + 3\,9 + 4\cdot9 + 5\cdot35 + 5\cdot65) + 0\cdot4(5\cdot3)}{8 + 0\cdot4}$$

$$= \frac{28\cdot3}{8\cdot4} = 3\cdot36,$$

$$\bar{y} = \frac{(1\cdot7 + 2\cdot55 + 3\cdot2 + 3\cdot6 + 3\cdot75 + 3\cdot5 + 2\cdot75 + 1\cdot7) + 0\cdot4(1\,1)}{8 + 0\cdot4}$$

$$= \frac{23\cdot2}{8\cdot4} = 2\cdot76.$$

**Work.**

The formula for the C. of G. is usefully applied to certain problems connected with work.

To find the work done in pumping the water in the given vessel to the level KL.

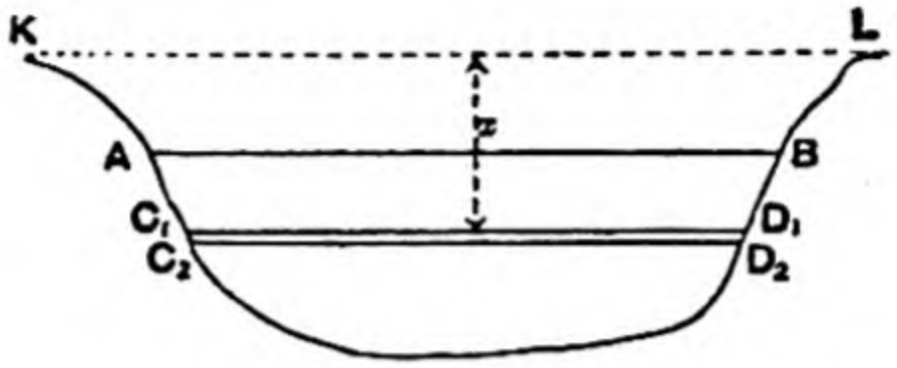



FIG. 270.

The water may be supposed to be removed in layers. Let $w_1$ be the weight of water in the thin layer $C_1D_1$, $C_2D_2$ at a distance $x_1$ from the level KL. The work done in raising this layer will be $x_1w_1$. The total work done will be $w_1x_1 + w_2x_2 + \ldots = \Sigma wx$. But if $\bar{x}$ be the depth of the C. of G. of the water,

$$\bar{x} = \frac{\Sigma wx}{\Sigma w};$$

$$\therefore \text{ work done} = \Sigma w \,.\, (\bar{x}),$$

which is equal to the work required to raise all the water from a depth equal to the depth of its C. of G.

## EXAMPLES XXXV.

**1.** Find graphically the C. of G. of the given figure. Check by calculation.

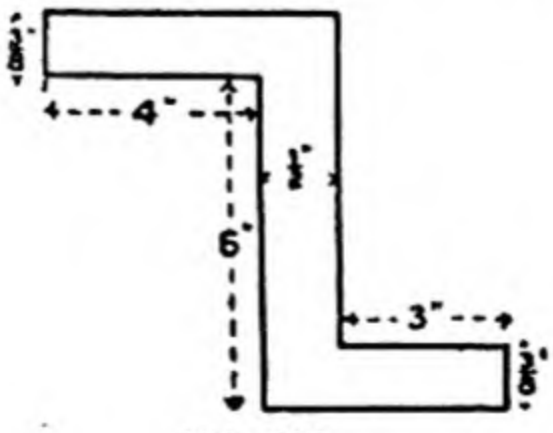



FIG. 271.

**2.** Plot the curve given by the points:

| $x$ | 0 | 1 | 4 | 6 | 7 | 3 | 10 | 11·5 | 15 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 38 | 38·5 | 38·5 | 35·5 | 32·5 | 27·5 | 22 | 19·0 | 15·7 | 11·0 | 3·85 |

Find graphically the coordinates of the centroid of the area between the curve, the $x$-axis, and the ordinates $y = 38$ and $y = 3{\cdot}85$. Check by calculation.

**3.** Plot a smooth curve through the following points:

| $x$ | 1 | 1·5 | 2 | 2·5 | 2·7 | 3·0 | 3·5 | 4 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 0·4 | 0·85 | 1·2 | 1·37 | 1·4 | 1·37 | 1·15 | 0·8 |

Find the coordinates of the C. of G. of the arc.

**4.** Join the points A(14), B(4, 5·5), C(6, 4·6), D(8·2), E(3, 1). Find the coordinates of the centroid of the area.

**5.** Plot a smooth curve through the points:

| $x$ | 0·3 | 0·5 | 0·8 | 1·2 | 1·8 | 2·0 | 2·2 | 2·5 | 3·0 | 3·5 | 4·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y_1$ | 2·0 | 1·5 | 1·2 | 1·1 | 1·0 | 0·9 | 0·8 | 0·5 | 0·2 | 0·3 | 2·0 |
| $y_2$ | 2·0 | 2·55 | 2·9 | 3·1 | 2·9 | 2·5 | 2·2 | 2·1 | 2·7 | 2·9 | 2·0 |

Find the centroid of the area.

**6.** A shaft is 200 ft. deep. Find the work done in bringing the earth to the surface, if a cubic foot weighs 140 lbs. Area of cross-section 120 sq. ft.

**7.** A solid circular cylinder rests with its circular base on a horizontal plane. Find the work done in tilting it until it is about to fall over. Diameter, 3 ft.; height, 8 ft.; weight, $\frac{1}{2}$ ton.

**8.** A uniform chain, 30 ft. long, weighing 10 lbs. per foot, hangs vertically. Find the work done in winding it up.

**9.** A rectangular tank, 6 ft. long, 3 ft. wide and 3 ft. deep, is filled from a cylindrical tank of radius 3 ft., the surface of the water being 10 ft. below the bottom of the empty tank. How much work is done in filling the tank, if 1 cu. ft. of water weighs 62·3 lbs.?

**10.** Find the number of foot-lbs. of work done in drawing up a Venetian blind consisting of 32 laths, each $\frac{1}{4}$ in. thick and of weight 4 oz. The highest lath is 2 in. from the top of the window, and the laths are 2 in. apart when the blind is down.

**To find the C. of G. of a tetrahedron.**

The tetrahedron may be divided into a series of laminae ($B_1C_1D_1$) parallel to the base.

Let $G_1$ be the C. of G. of the $\triangle$ BCD. Join $AG_1$.

Let $AG_1$ cut the plane $B_1C_1D_1$ in K, which will lie in the plane ABE.

By similar triangles, $B_1K : B_1E_1 = BG_1 : BE$, so that K is the C. of G. of the lamina $B_1C_1D_1$.

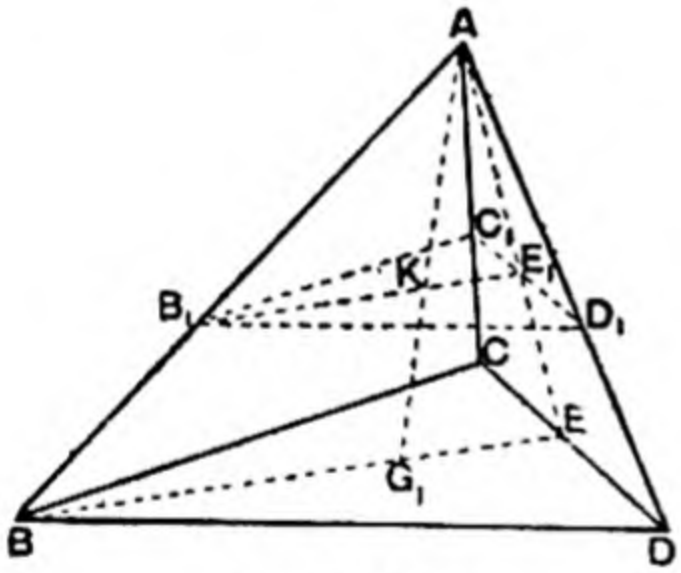



FIG. 272 (*a*).

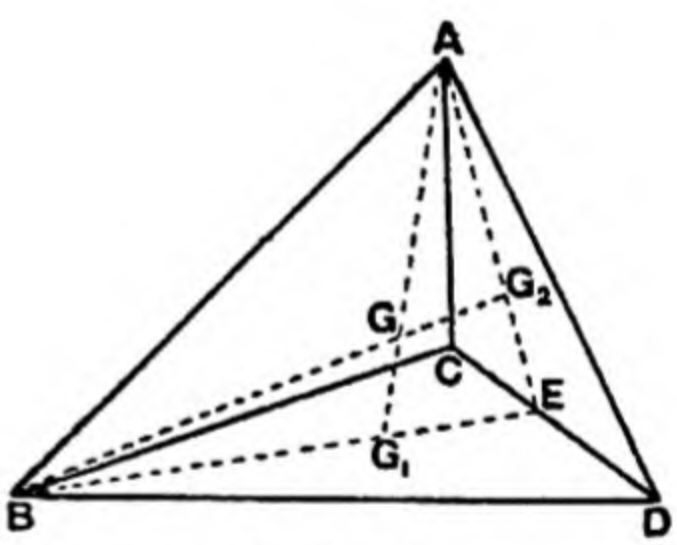



FIG. 272 (*b*).

Similarly, the C. of G. of each strip lies in $AG_1$.

Again, the tetrahedron can be divided into laminae parallel to ACD, so that the C. of G. of the tetrahedron will lie in $BG_2$.

$BG_2$ and $AG_1$ both lie in the plane ABE. Let them intersect in G.

By similar triangles ABG and $G_1GG_2$, we have

$$G_1G : GA = G_1G_2 : AB ;$$

$$\therefore\ G_1G = \tfrac{1}{3}GA ; \quad \therefore\ G_1G = \tfrac{1}{4}G_1A.$$

Therefore the C. of G. of the pyramid is in the line joining A to the C. of G. of the base at a point distant $\frac{1}{4}$ of $AG_1$ from $G_1$.

It will be clear that this is the same point as the C. of G. of four equal weights placed at the angular points of the figure.

**C. of G. of a Cone.**

Suppose a regular polygon to be inscribed in the circular base. Join the angular points to B, the C. of G. of the base.

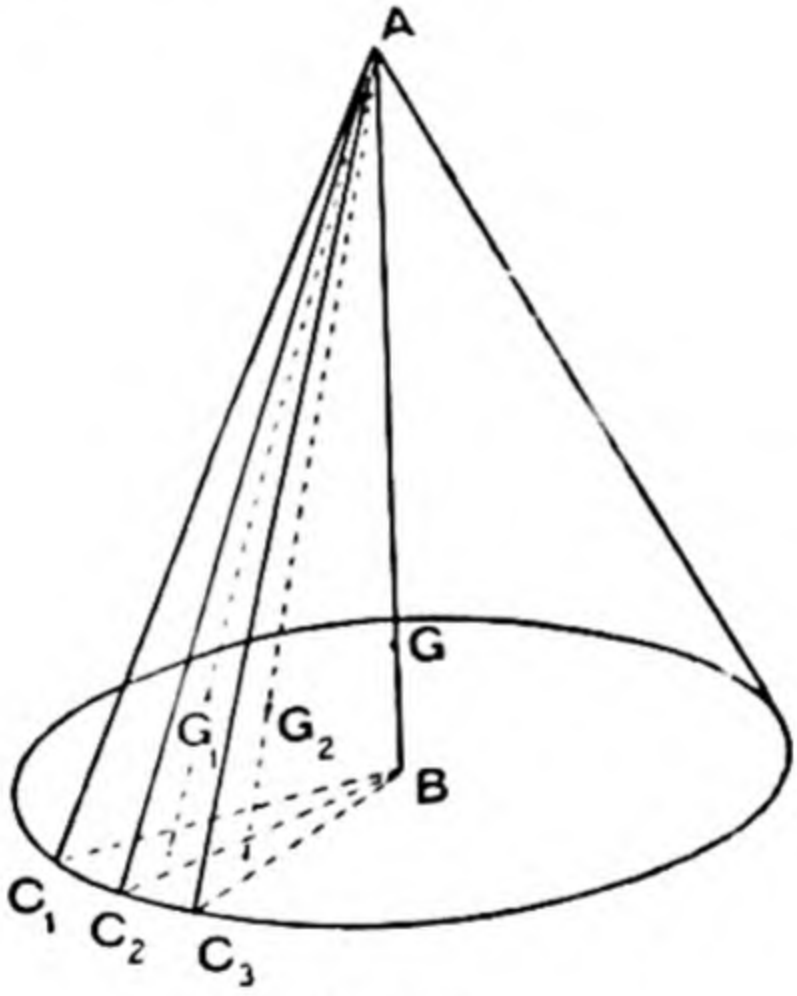



FIG. 273.

A series of pyramids is formed, each of which has its C. of G. on a plane parallel to the base, $\frac{1}{4}$ of the altitude from the base.

The mass of each pyramid is the same, so that we may suppose equal weights to act at the points $G_1$, $G_2$, etc.

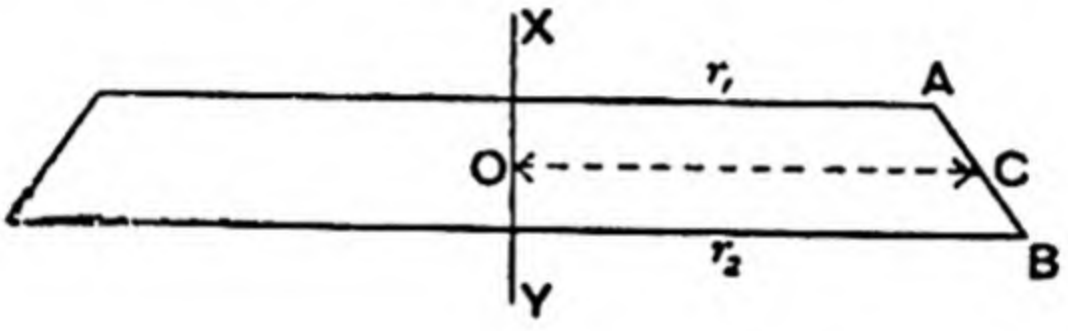
If the number of sides of the polygon is indefinitely increased, the solid so formed approximates to a cone, and

the resultant of the weights at $G_1$, $G_2$, etc., will be at the centre of the circle on which $G_1$, $G_2$, etc. lie, *i.e.* on the line AB, $\frac{1}{4}$ of its distance from B.

The student who is likely to proceed to the study of the Calculus should omit the following theorems:

**Theorems of Guldin or Pappus.**

I. The surface of a solid formed by revolving a curve about a straight line in its plane equals *Length of curve* × *circumference of the circle traced out by the C. of G. of the curve.* A solid so formed is called a *Solid of Revolution.*



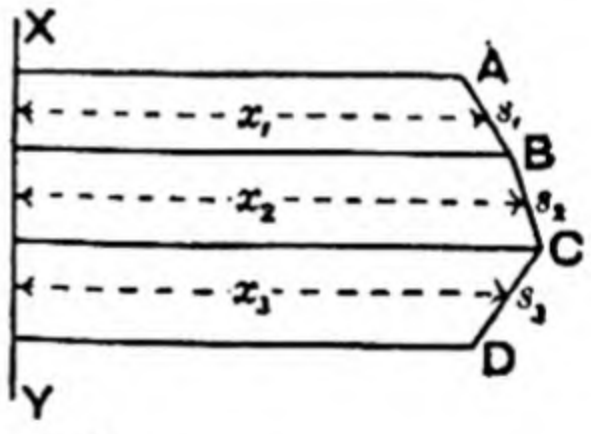




FIG. 274.

Consider the line AB rotating about the axis XY. It will trace out the frustum of a cone. The surface of the frustum of a cone is

$$2\pi\left(\frac{r_1+r_2}{2}\right).\text{AB} \text{ or } 2\pi(\text{OC}).\text{AB}.$$

If the broken line ABCD revolves round XY, the surface of the figure generated will be

$$s_1 2\pi x_1 + s_2 . 2\pi x_2 + s_3 2\pi x_3 + \ldots = 2\pi[s_1 x_1 + s_2 x_2 + s_3 x_3 + \ldots].$$

But if $\bar{x}$ be the distance of the c. of G. of AB, BC, CD, ... from XY, we have

$$\bar{x} . \Sigma s = \Sigma s x ;$$

$$\therefore \text{ area} = 2\pi\bar{x} . \Sigma s$$

$$= \Sigma s . (\text{path of c. of G.}).$$

The proposition will hold good however small the parts $s_1$, $s_2$, ... may be, and therefore is true for a curve.

II. The *volume* of a solid of revolution formed by rotating a plane area about a line in its plane equals *Area* × *circumference of the circle traced out by its C. of G.*

Consider the solid traced out by the revolution of the rectangle ABCD about XY parallel to the side AB.

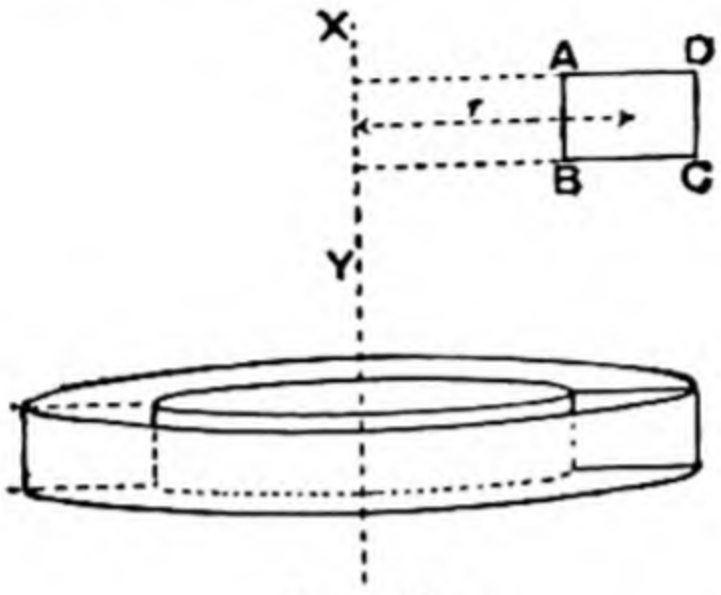



FIG. 275.

Let $b$ be the breadth (AD) of the rectangle and $r$ the distance of its C. of G. from the axis.

$$\text{Volume of solid formed} = \pi\left(r+\frac{b}{2}\right)^2 . \mathsf{AB} - \pi\left(r-\frac{b}{2}\right)^2 \mathsf{AB}$$
$$= \pi \mathsf{AB} . 2br$$
$$= (\mathsf{AB} . b)(2\pi r)$$
$$= \text{area} \times \text{path of C. of G.}$$

Now, if a series of rectangles revolve round the axis, we have, if their areas are $\mathsf{A}_1$, $\mathsf{A}_2$, ..., and their C. of G. respectively $x_1$, $x_2$ from the axis,

$$\text{Volume of solid} = \mathsf{A}_1 2\pi x_1 + \mathsf{A}_2 2\pi x_2 + \cdots$$
$$= 2\pi[\mathsf{A}_1 x_1 + \mathsf{A}_2 x_2 + \cdots].$$

If $\bar{x}$ is the distance from XY of the C. of G. of the whole area, and A the total area, we have

$$\bar{x} = \frac{\Sigma \mathsf{A}x}{\Sigma \mathsf{A}};$$

$$\therefore \text{ volume of solid} = 2\pi\bar{x} . (\Sigma \mathsf{A}) = \text{area} \times \text{path of C. of G.}$$

Since an area of any shape may be considered as being made up of an infinite number of very small rectangles, the theorem holds good for a solid formed by the revolution of a figure of any shape.

**Surface of a Sphere.**

Consider the surface formed by the rotation about AO of the chord PQ of a circle.

$$\text{Area} = PQ \,.\, 2\pi \,.\, (RC)$$
$$= PQ \,.\, 2\pi \,.\, OR \,.\, \sin\theta = AB \,.\, (2\pi \,.\, OR).$$

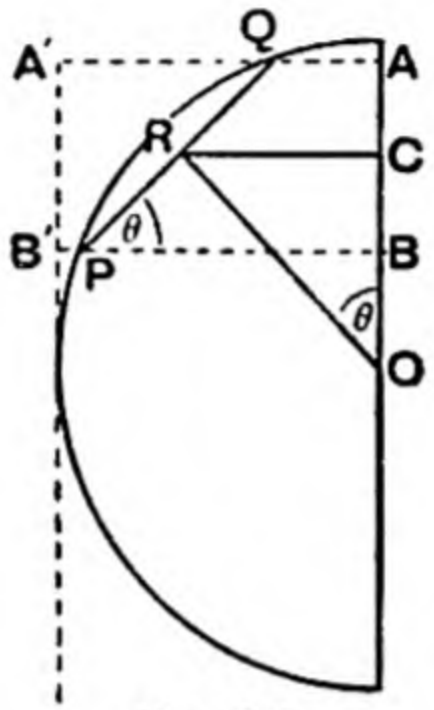



FIG. 276.

This will be true no matter how small we make PQ, and will be true for each element of surface so formed. As the chords are indefinitely diminished, OR becomes more and more nearly equal to $r$; $\therefore$ area $= 2\pi r(\Sigma AB)$.

It follows therefore that the area of a belt on a sphere equals the area cut off by the parallel planes bounding the belt, on a circumscribing cylinder whose axis is at right angles to these planes.

Hence, Surface of whole sphere $= 2\pi r \,.\, (2r) = 4\pi r^2$.

**C. of G. of Surface of a Hemisphere.**

If we divide up the surface of a hemisphere by equidistant planes parallel to the plane face, the area of the surface of each belt equals the area of the corresponding section on the circumscribing cylinder, so that the C. of G. of the surface of the hemisphere coincides with the C. of G. of the circumscribing cylinder, and is therefore $\frac{r}{2}$ from the centre along the axis.

**C. of G. of a Solid Hemisphere.**

A solid hemisphere may be considered as made up of a large number of equal small cones whose bases make up the surface of

the hemisphere, each having its vertex at the centre of the circle which forms the plane face of the hemisphere.

The c. of g. of each cone will lie on a hemispherical shell, whose radius will be approximately $\frac{3}{4}r$, and this radius will become more and more nearly equal to $\frac{3r}{4}$ as the number of cones is indefinitely increased, so that their sum approximates more and more closely to the volume of the hemisphere.

The c. of g. of this hemispherical shell then coincides with the c. of g. of the solid hemisphere, whose distance from the centre of the plane face is therefore $\frac{1}{2} \times \frac{3r}{4} = \frac{3r}{8}$.

### C. of G. of a Semi-circular Arc.

If a semi-circular wire ACB rotates about the diameter AB, it will trace out the surface of a sphere.

By Guldin's Theorem, we have

Area of sphere = arc ACB × path of c. of g. ;

$$\therefore \ 4\pi r^2 = \pi r \times 2\pi \bar{x} \ ;$$

$$\therefore \ \bar{x} = \frac{2r}{\pi}.$$

### C. of G. of a Semi-circle.

If a semi-circle ACB rotates about the diameter AB, it will generate a sphere.

By Guldin's Theorem,

Vol. of sphere = area ACB × path of c. of g. ;

$$\therefore \ \frac{4}{3}\pi r^3 = \frac{\pi r^2}{2} \times 2\pi \bar{x} \ ;$$

$$\therefore \ \bar{x} = \frac{4r}{3\pi}.$$

## EXAMPLES XXXVI.

**1.** A solid figure consists of a right circular cone and a hemisphere of the same material and diameter joined together. If the height of the cone equals the radius of the hemisphere, find the c. of g.

**2.** The base of a solid cylinder, radius 6 in., is fastened to the circular end of a solid hemisphere of the same radius. What is the greatest length of the cylinder, if the solid is to be in stable equilibrium when it rests with its curved surface on a horizontal plane ?

**3.** Find the C. of G. of a hemispherical shell of uniform material. External radius 6 in., thickness 2 in.

**4.** A cube, whose edge is of length 1 ft., has a hemispherical cavity of radius 3 in. cut in the centre of one of its faces. Find the C. of G. of the remainder.

**5.** A semi-circular wire of radius 1 ft. is suspended freely from one end. Find the angle the line joining the two ends will make with the vertical.

**6.** Prove that the distance from the centre of the C. of G. of an arc which subtends an angle $\alpha$ at the centre is $\dfrac{2r \sin \frac{\alpha}{2}}{\alpha}$.

(Note that if the arc revolves round a boundary radius, it traces out the surface of a spherical cap.)

**7.** A piece of wire 1 ft. long is bent into a circular arc, so that it subtends an angle of 60° at the centre of the circle. Find its C. of G.

**8.** A thin hemispherical bowl has a circular lid of the same material. Find the distance of the C. of G. of the whole from the centre of the lid when the lid is shut.

**9.** A hollow vessel is made of a hemisphere with a top the shape of a right cone of the same material. If the figure will rest in any position when the hemispherical part is resting on a horizontal plane, find the semi-vertical angle of the cone.

**10.** A uniform wire is bent into the shape of a circular arc and the two bounding radii. Find the angle subtended by the arc at the centre, in order that the C. of G. of the wire may coincide with the centre of the circle of which the arc forms a part.

**11.** A thin wire 12 ft. long is coiled up in a circle of radius 6 in. Find the distance of the C. of G. of the coil from the centre of the circle.

**12.** A uniform thin wire is bent into the shape of an **S** formed by two semi-circles, one of radius 6 in., the other of radius 1 ft. Find the coordinates of the C. of G. referred to two axes, one passing through the centres of the semi-circles, the other at right angles to this line through the junction of the two arcs.

# CHAPTER XII.

## FORCES IN THREE DIMENSIONS.

THE two following propositions dealing with forces in three dimensions are immediately deduced from the parallelogram of forces:

(i) If three forces X, Y, Z acting at a point O are represented in magnitude and direction by the adjacent edges of a parallelepiped, then the diagonal of the figure passing through O represents the resultant of the three forces in magnitude and direction.

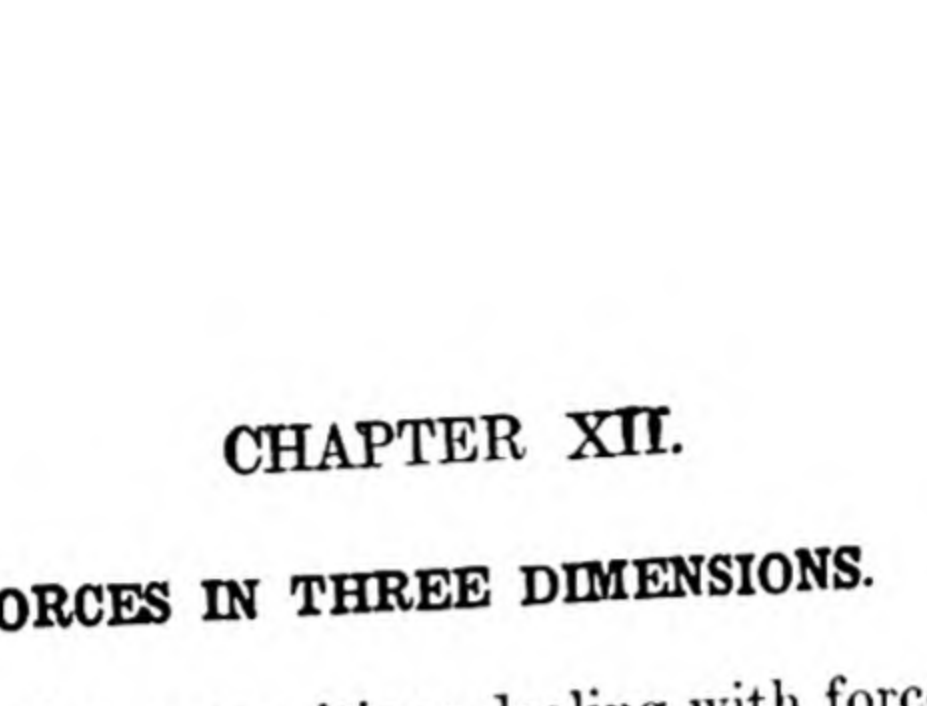



FIG. 277.

The resultant of X and Y acting at O is represented by OD, the diagonal of the parallelogram OADB.

The resultant of OD and Z is represented by OE, which is the diagonal of the parallelepiped.

(ii) A force R acting at O may be resolved into components along any three given lines through O not in one plane. This can be done in only one way, since there is only one parallelepiped with OE as diagonal, having its adjacent edges along OA, OB, OC.

It is usual for the lines OA, OB, OC to be at right angles to each other.

Since $R^2 = X^2 + Y^2 + Z^2$, it follows that if $R = 0$, then $X = 0$, $Y = 0$ and $Z = 0$.

**Couples in Three Dimensions.**

In order to obtain the equations which determine the conditions of equilibrium of a rigid body acted upon by forces in three dimensions, it is necessary to show how couples which act in non-parallel planes may be compounded.

In order to determine a couple completely, we must know:

(1) Its moment, (2) the direction of the plane in which it acts, (3) the sense in which rotation is produced.

From any point A in the plane of the couple, draw a straight line AB perpendicular to the plane.

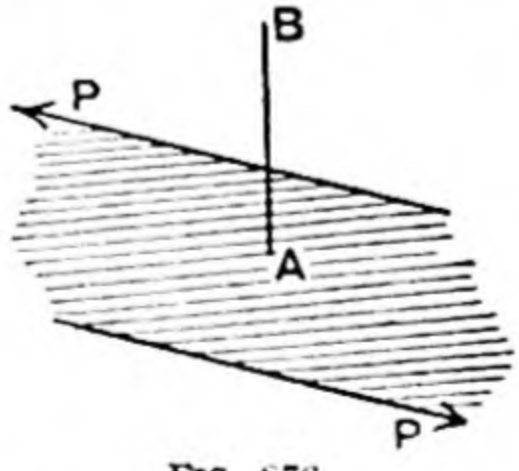



FIG. 273.

This line completely determines the couple, if we make the following conventions:

(1) The length AB represents, according to some scale, the number of units in the moment of the couple.
(2) The plane of the couple will be perpendicular to the line.
(3) The sense of the couple is determined by the direction of the line AB, which is drawn so that the direction of AB and the rotation due to the couple form a right-handed screw.

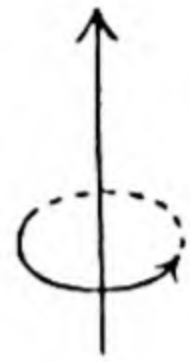


FIG. 279.

The line AB is called the axis of the couple or the couple vector.

If the couple in Fig. 278 were reversed, the line AB would be drawn on the other side of the plane.

**To find the Resultant of Two Couples acting in Non-parallel Planes.**

The forces of one of the couples may be changed until the arm is equal to the arm of the other couple, provided the moment of the couple remains unaltered. (Chapter VIII., p. 193.)

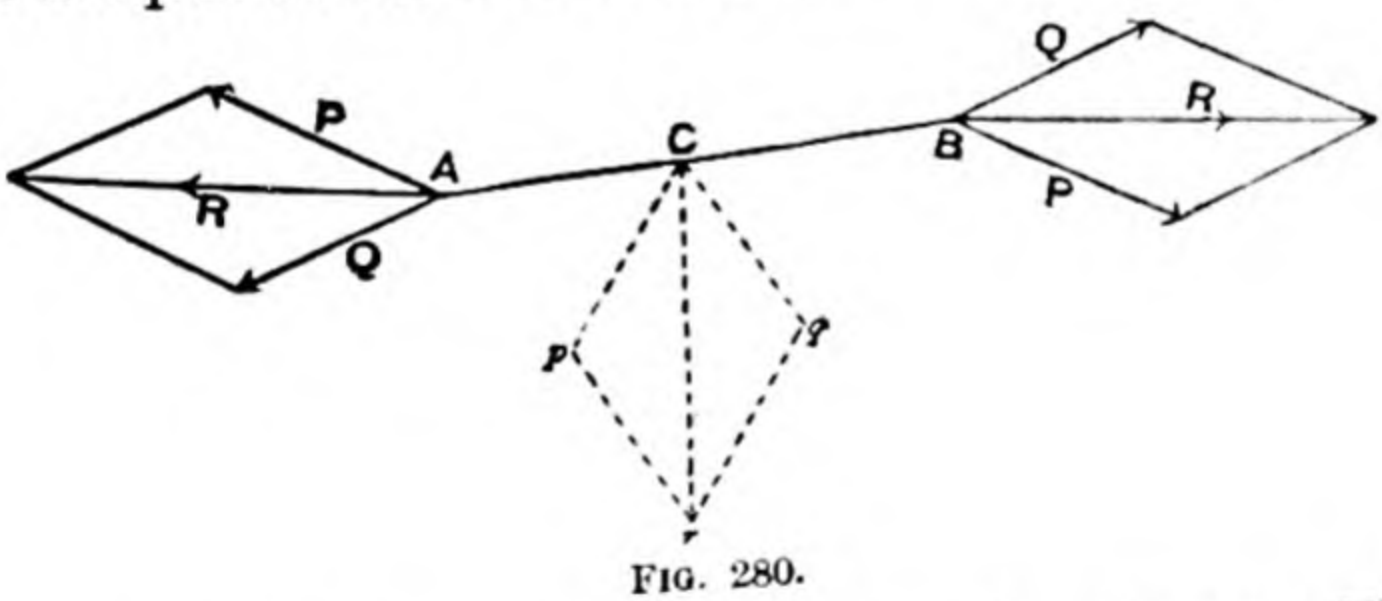



FIG. 280.

Let the couples now be P(AB) and Q(AB).

They may be moved, each parallel to itself, until they have a common arm AB lying along the intersection of the two planes in which the couples act. as in Fig. 280 (see

p. 194). The forces P and Q at A are equivalent to a force R, found by the parallelogram of forces.

Similarly for the forces at B.

The forces (R, R) at A and B are equal and parallel, so that the two given couples are equivalent to the couple R(AB).

From any point C in AB, draw the couple vectors C$p$ and C$q$; C$p$ perpendicular to the plane of the couple P(AB) and equal to P(AB), forming with the couple a right-handed screw.

Similarly for C$q$.

If we compound C$p$, C$q$ by the parallelogram law, we get C$r$ perpendicular to the plane of the couple R(AB). The line C$r$ will be equal to the moment R(AB), since the sides of the parallelogram C$prq$ are respectively perpendicular to the sides of the parallelogram whose sides are P, Q with diagonal R, and equal to P(AB), Q(AB) respectively.

C$r$ therefore represents the axis or couple vector of a couple which is equivalent to the two given couples whose vectors are C$p$ and C$q$. Couples may therefore be compounded by the parallelogram law, and consequently couples represented by their vectors may be resolved into components in any three directions mutually at right angles.

If, for instance, C$p$ and C$q$ were at right angles,

$$Cr^2 = Cp^2 + Cq^2.$$

If the resultant couple were zero, we should have $Cp = 0$ and $Cq = 0$.

**General Conditions of Equilibrium.**

Let OX, OY, OZ be three axes of coordinates at right angles to one another.

A force P acting at the point A, whose coordinates are $x$(OM), $y$(MN), $z$(NA), can be resolved into three components $P_x$, $P_y$, $P_z$, parallel to the axes.

Suppose $P_z$ to act at N, and introduce at M and at O two pairs of opposite forces, all of magnitude $P_z$.

The five forces ($P_z$) may now be considered as a couple $P_z$(NM), a couple $P_z$(MO) and a single force $P_z$ acting at O.

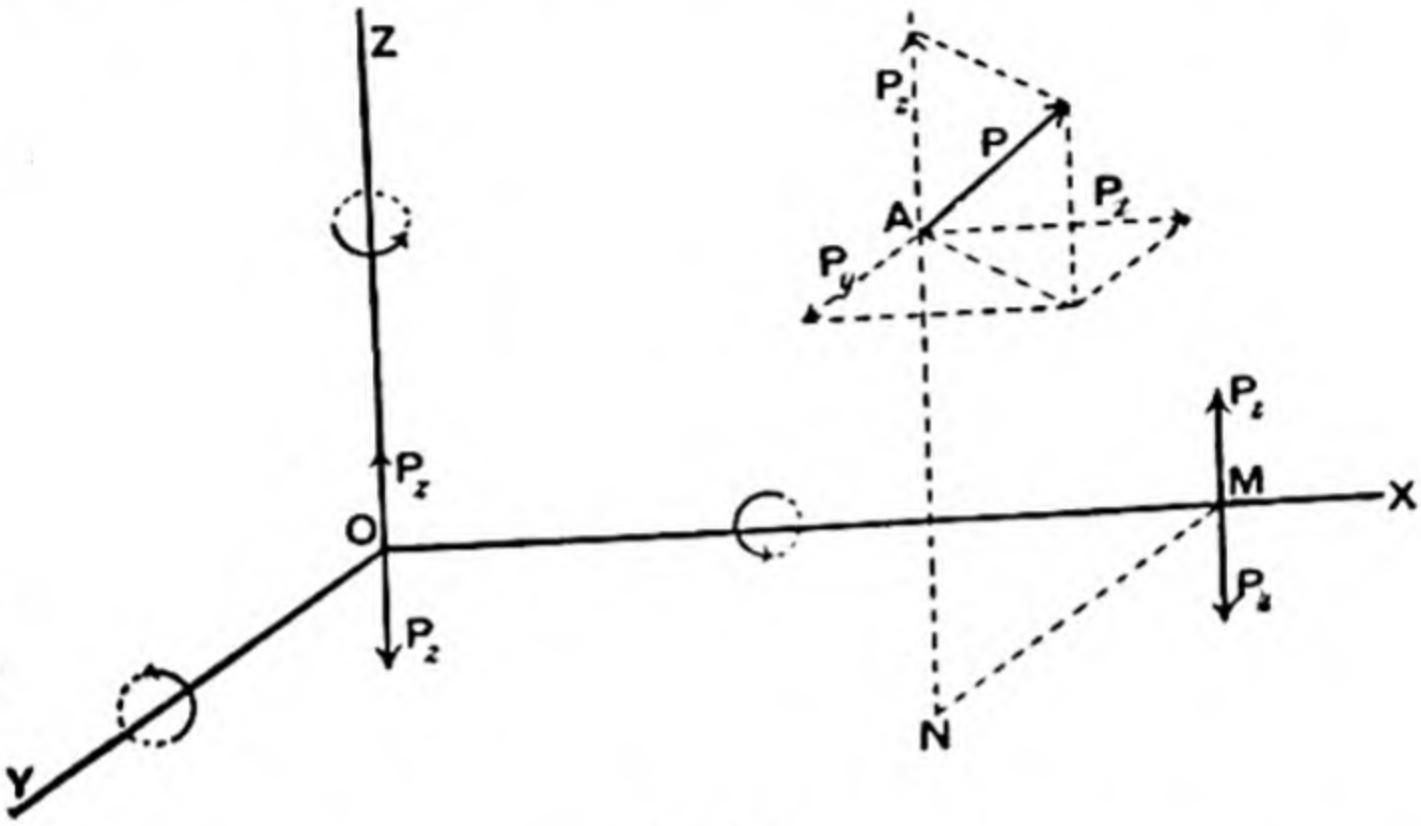



FIG. 281.

Taking the couples as positive, if they form a right-handed screw with the positive direction of the axes, these couples are $-P_z(y)$ about OX, $+P_z(x)$ about OY.

If the other components are treated in a similar way, we get

$P_x$ acting at O, together with the couples
$-P_x(z)$ about OY and $+P_x(y)$ about OZ;
$P_y$ acting at O, together with the couples
$-P_y(x)$ about OZ and $+P_y(z)$ about OX.

The force P at A is therefore equivalent to $P_x$, $P_y$, $P_z$ acting at O, together with the couples

$zP_y - yP_z$ about OX,
$xP_z - zP_x$ about OY,
$yP_x - xP_y$ about OZ.

Each force acting at the various points on the rigid body may be treated in a similar way, and the whole system will then be equivalent to

(1) $X = \Sigma P_x$, the sum of all the components along OX.

(2) $Y = \Sigma P_y$, " " "

(3) $Z = \Sigma P_z$, " " "

These may all be compounded into a single force P acting at O.

There will also be a system of couples

(4) $L = \Sigma(zP_y - yP_z)$ about OX.

(5) $M = \Sigma(xP_z - zP_x)$ about OY.

(6) $N = \Sigma(yP_x - xP_y)$ about OZ.

These may be represented by their couple vectors along the three axes and compounded into a single couple C.

For equilibrium $R = 0$ and $C = 0$,

*i.e.* $X = 0$, $Y = 0$, $Z = 0$, $L = 0$, $M = 0$, $N = 0$.

This result is equivalent to stating that for equilibrium there must be no resultant forces along OX, OY or OZ ; that is to say, the body cannot be translated in any direction. There can also be no moment about any of the three axes, so that the body cannot be rotated in any direction.

EXAMPLE 1. A circular drum, which can rotate freely round its axis, is acted upon by a couple of moment C, the plane of

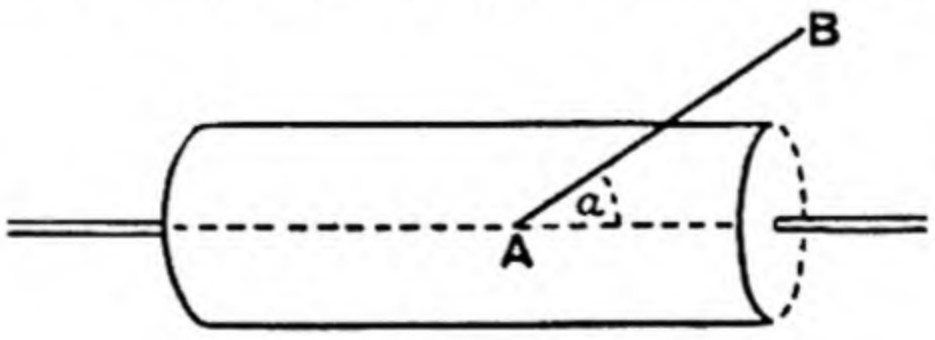



FIG. 282.

which makes an angle $a$ with the section of the drum which is perpendicular to the axis. To find the moment of the couple which is effective in rotating the drum.

The couple vector AB will make an angle $\alpha$ with the axis of the drum, and it may be transferred so that A is on the axis. It may be resolved into two couple vectors, AB cos $\alpha$ along the axis and AB sin $\alpha$ perpendicular to the axis. The forces represented by the latter will be in the same plane as the axis, or parallel to it, and will produce no rotation. The forces of the couple vector AB cos $\alpha$ will produce a rotation equivalent to that due to a couple of moment C cos $\alpha$.

EXAMPLE 2. The tail board of a trap weighs 40 lbs., and is let down by two chains fastened to B and C, making 45° with the horizontal. If one chain breaks, find the reactions at the hinges A and D. AD = 4 ft., AB = 3 ft.

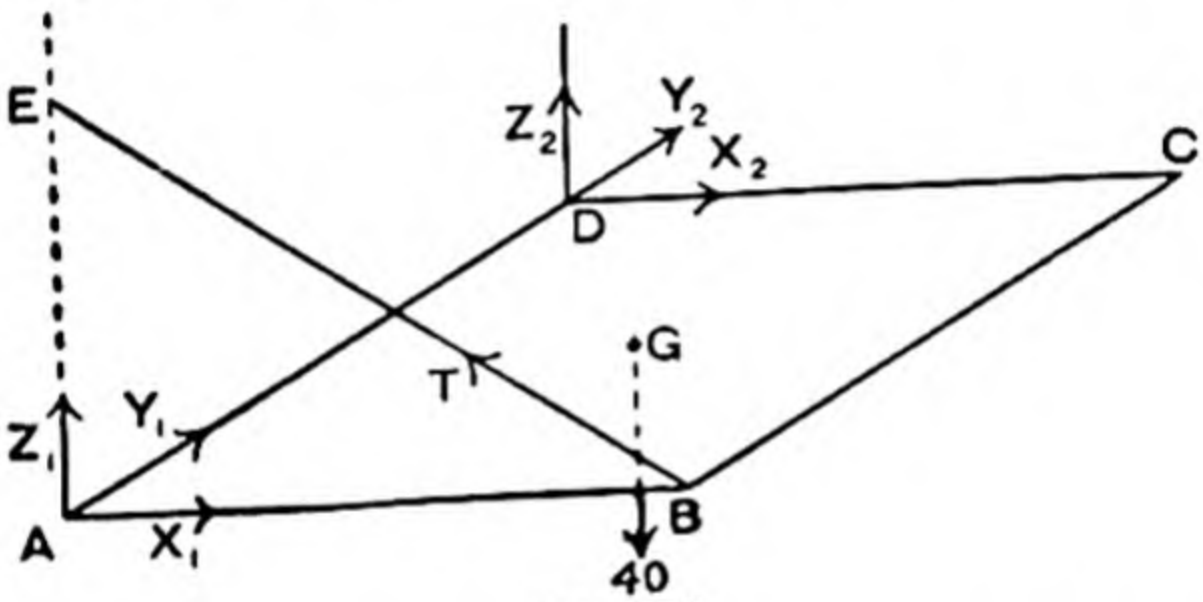



FIG. 283.

Take A as origin, AB as axis of X, AD as axis of Y and AE vertical as axis of Z.

Let T be the tension in the chain EB; $X_1, Y_1, Z_1$ the components of the reaction at A; $X_2, Y_2, Z_2$ the components at D.

Resolve T into T cos 45 along BA, and T sin 45 vertical.

We have then, by resolving parallel to the axes,

(i) $X_1 + X_2 - T\cos 45 = 0.$

(ii) $Y_1 + Y_2 = 0.$

(iii) $Z_1 + Z_2 + T\sin 45 - 40 = 0.$

Taking moments about the axes, we have

(iv) $4Z_2 - 40(2) = 0$ about AB.

(v) $T\sin 45(3) - 40(\frac{3}{2}) = 0$ about AD.

(vi) $4X_2 = 0$ about AE.

From (iv), $Z_2 = 20$; from (v), $T = 20\sqrt{2}$; from (vi), $X_2 = 0$;

„ (i), $X_1 = 20$; „ (iii), $Z_1 = 0$; „ (ii), $Y_1 + Y_2 = 0$,

but both will equal zero, if we assume that the hinges allow motion freely along AD.

EXAMPLE 3. A weight W rests on a rough inclined plane of angle $\alpha$, and is fastened by a string to a point A on the plane. If $\lambda$ be the angle of friction ($\lambda < \alpha$), find the greatest angle the string can make with a line of greatest slope.

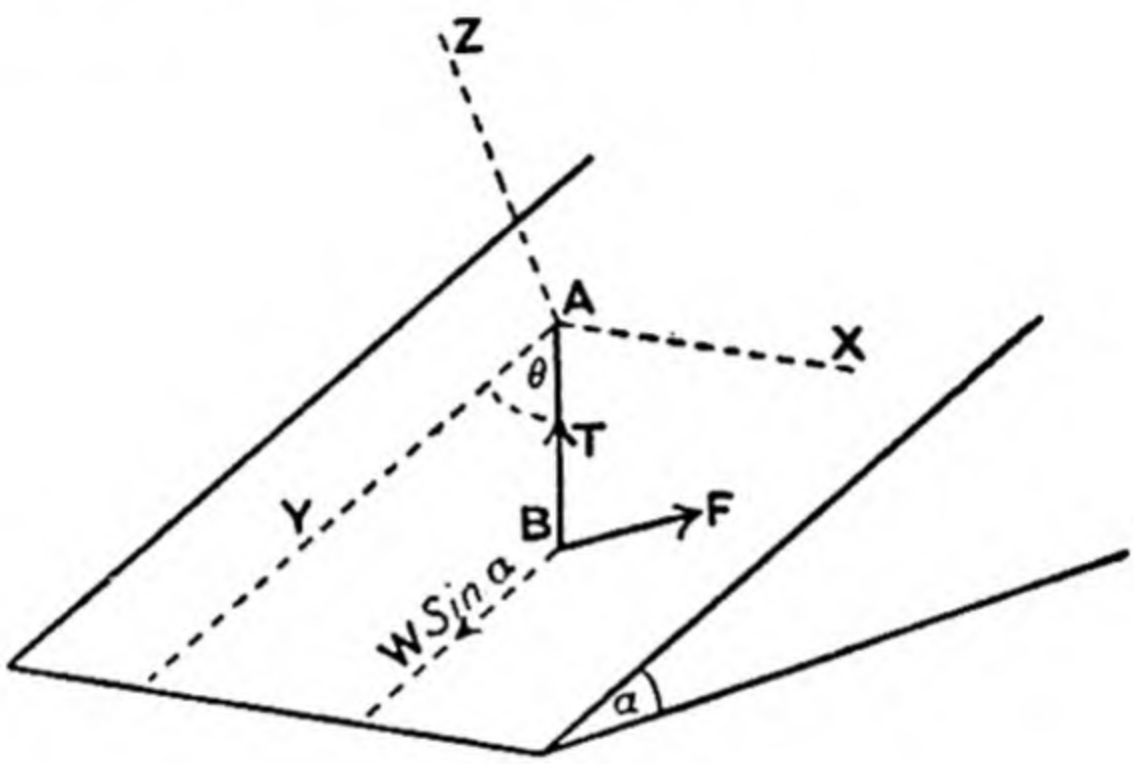



FIG. 284.

Resolve W into $W\cos\alpha$ at right angles to the plane, and $W\sin\alpha$ along a line of greatest slope. The weight at B is on the point of moving along the arc of a circle, so that $F = \mu N$ will act at right angles to AB.

Take for axes AY along the line of greatest slope, AX at right angles to it on the plane and AZ perpendicular to the plane.

Resolve T into $T\cos\theta$ parallel to AY and $T\sin\theta$ parallel to AX.

We have then the following equations:

$$T\sin\theta = F\cos\theta, \quad T\cos\theta + F\sin\theta = W\sin\alpha$$

$$N = W\cos\alpha, \qquad F = \mu N,$$

from which $$\sin\theta = \tan\lambda\cot\alpha.$$

## EXAMPLES XXXVII.

**1.** The lid ABCD of a box is turned through 180°, and is kept horizontal by a vertical force at C. If the lid weighs 20 lbs., and is hinged at E and F in the edge AB, find the reactions at these hinges. AB = 3 ft., BC = 2 ft., AE = BF = 3 in.

**2.** A door ABCD, 6 ft. high, 4 ft. wide, weighing 80 lbs., turns freely on hinges at E and F in AB, of which the lower carries all the weight. A handle in the middle of the door, 2 ft. from each side, is pulled with a force of 20 lbs. at right angles to the door, which is kept in equilibrium by a parallel force P on the other side at G in CD, 2 ft. from the top D. If AE = BF = 6 in., find P and the components of the reactions at the hinges.

**3.** A square trap door ABCD, hinged at A and B, weighs 30 lbs., and is kept in a horizontal position by a rope fastened to C and to a point E vertically above the opposite corner A, so that CE makes 45° with the horizontal diagonal AC. Find the reactions at the hinges, if AB = BC = 4 ft.

**4.** A table ABCD, 4 ft. square, is supported by a stand in the middle having three legs, whose feet form an equilateral triangle of side 2 ft., one side of which is parallel to AB. The table is uniform and weighs 50 lbs.; find the vertical reactions at the feet when a weight of 10 lbs. is placed at C.

**5.** A rectangular bracket ABCD, having a triangular piece ABE at right angles to it at one end, is supported from nails at A and B and touches the wall at E. Neglecting the weight of the bracket, find the reaction at E and the components of the reactions at A and B, when a weight of 2 lbs. is placed on the bracket at the corner C. AB = 8 in., BC = 12 in., AE = BE = 5 in.

**6.** A vertical bar AB is supported by three legs, whose feet form an equilateral triangle of side 2 ft. A horizontal rod, parallel to one of the sides of the equilateral triangle, is screwed into the top of the vertical bar, and when tight is twisted by a couple of 50 lbs.-ft. acting in a vertical plane at right angles to the rod. If the total weight of the tripod is 150 lbs. and acts along AB, find the pressure of the feet on the ground for both directions of rotation of the couple.

**7.** A rectangular window ABCD, weighing 10 lbs., which turns on hinges at E and F in the vertical edge AB, of which the lower F carries all the weight, is kept open by a horizontal bar fastened to the bottom corner C and to the corresponding corner G of the frame, so that GBC is an isosceles triangle and $G\hat{B}C = 40°$. Find the tension in the bar and the components of the reactions at the hinges in the plane of the window and at right angles to it, when the window is pulled by a force of 5 lbs. (at right angles to the plane of the window on the opposite side from CG), which acts at K on the edge CD, 3 ft. from C. AB = 4 ft., BC = 2 ft., BF = AE = 6 in.

**8.** ABCD is the lid of a box which is hinged at A and at B and weighs 12 lbs. It is propped open at an angle of 30° to the horizon by a stick at C, which rests on the corresponding corner of the box. If AB = 4 ft., BC = 3 ft., find the thrust in the stick and the components of the reactions at the hinges.

**9.** The axle shown in the figure is 10 in. in diameter, the wheel is 40 in. in diameter, and the whole is supported in smooth bearings 50 in. apart. A rope wrapped round the axle carries a load of 1200 lbs. in a plane distant 15 in. from the right-hand

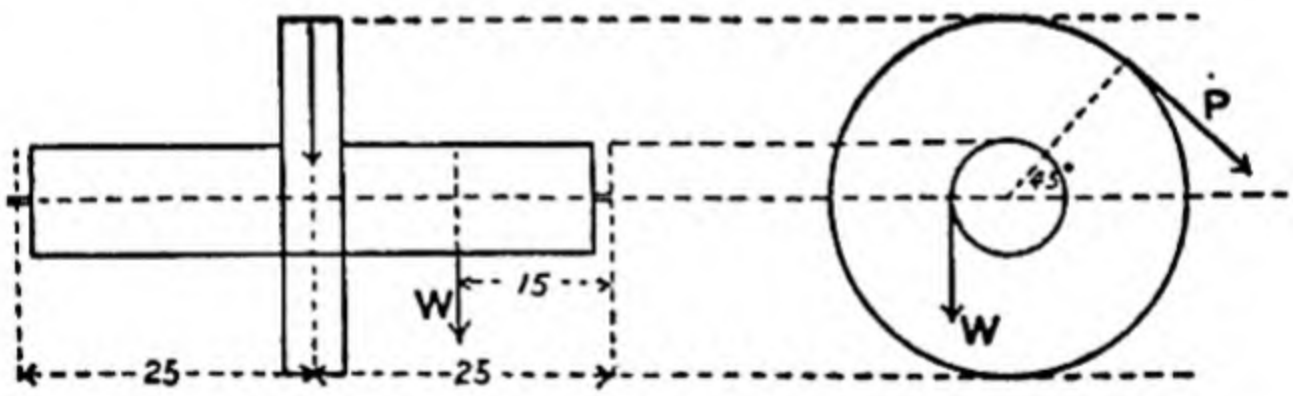



FIG. 285.

bearing, and this is supported by a force P applied tangentially to the wheel at the end of a radius which makes 45° with the horizontal. Find the value of P, and the horizontal and vertical components of the force on each bearing. (C.U.)

**10.** When a gate is shut, the line of hinges is inclined to the vertical in the plane of the gate at 10°. The gate weighs 80 lbs. and is opened through an angle of 30°. If the perpendicular distance of the C. of G. of the gate from the line of hinges is 3 ft., find the friction couple at the hinges which is required to keep the gate in this position.

**11.** Three equal light rods OA, OB, OC are hinged together at O, and the other ends are fixed to a horizontal plane, so as to form a pyramid whose altitude is 4 ft. and base an equilateral triangle ABC of side 4 ft. A horizontal force of 10 lbs. is applied at O, parallel to a line AD, passing from A to D in BC, so that DAB = 45°. Find the vertical components of the forces at A, B and C.

**12.** The barrel of a windlass of radius 6″, and length 4 ft., is turned by two equal forces P applied tangentially, 6″ from the arm, to the handles, which project at right angles to the two arms, one at each end of the barrel. The rope supporting a load of 3 cwts hangs from the middle of the barrel. Neglecting the weight of the barrel and handles, find the components of the reactions at the supports when the handles, which are 2 ft. long, are inclined at 40° to the vertical.

**13.** A pole AB, 6 ft. long, smoothly hinged at one end A and lying across a rough horizontal rail at B, 2 ft. from A, is inclined at 30° to the horizontal when perpendicular to the rail. If $\mu = \frac{1}{2}$, show that, when the pole is turned round A, it will remain in equilibrium until the point of contact with the rail is $2\sqrt{3}$ ft. from B.

**14.** A uniform rod of weight W can turn freely about a fixed hinge at one end, and rests with the other end in contact with a rough vertical wall, making an angle $\alpha$ with the wall. Show that this end may rest anywhere on the arc of a circle subtending an angle $2\theta$ at its centre, where $\tan\theta = \mu \tan\alpha$. Find the pressure on the wall in either of the extreme positions.

**15.** Two equal forces P act on a cube of edge $a$, whose centre is fixed, along diagonals which do not meet, of two adjacent faces. Find the couple that will keep the cube at rest. (Two solutions.)

## MISCELLANEOUS EXAMPLES.

### I.

**1.** A circular disc of radius $r$ and weight $w$ is placed inside a smooth sphere of radius R ($>r$), and at a distance $d$ from the centre of the disc is fixed a weight W. Show that in the position of equilibrium the plane of the disc makes an angle with the horizontal whose tangent is $\dfrac{Wd}{(W+w)\sqrt{R^2-r^2}}$.

**2.** The figure shows the mechanism for supporting a lever lamp. AB, CE, EF are brass tubes, CE and EF being freely jointed at E. AE can turn freely in a vertical plane about D, the end of a fixed tube OD. M is a heavy circular disc of diam.

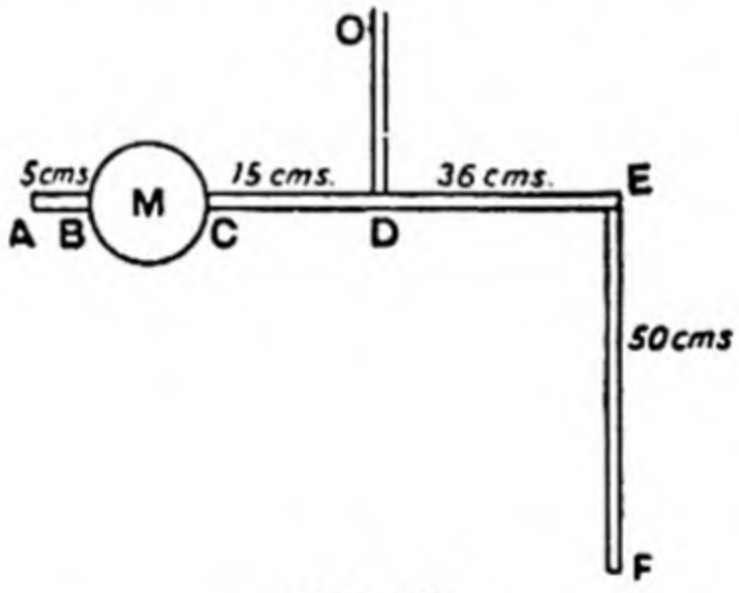



FIG. 286

10 cms., weighing 1 Kg., serving to counterpoise a lamp at F. The tubes are uniform, and weigh 300 grams per metre. Find to the nearest mm. the distance from D of the C. of G. of the portion ABMCD. Find also the weight of the lamp necessary for equilibrium in any position of the mechanism.

(C.S.C.)

**3.** The weight of a four-wheeled truck is 5 tons, and its C. of G. is midway between the front and back axles, which are 10 ft. apart. The truck is pulled by a horizontal force 3 ft. above rail level. The friction at the axles may be

neglected. Calculate the pull required to move the truck (*a*) with all wheels locked, (*b*) only front wheels locked, (*c*) only rear wheels locked. Take $\mu = 0{\cdot}4$.

**4.** Three uniform bars are hinged together at their ends so as to make a frame in the form of an isosceles triangle, which is suspended by its vertex from a fixed point. The weight of each side is W, the weight of the base $w$, and the angle at the vertex is $\beta$. Find the force between the rods at each of the lower hinges, and show that its direction is inclined to the vertical at an angle whose tangent is

$$\left(\frac{W}{w} + 1\right) \tan \frac{\beta}{2}.$$

**5.** ABCD is a rectangular door working in two vertical guides AB, DC. The weight of the door is 0·72 tons and AB = 6 ft. The door is raised by a vertical force of 0·8 ton in the plane of the door, acting 9″ from the centre, which tilts the door slightly so that contact with the guides occurs only at A and C. Find the reactions at these points and calculate $\mu$.

## II.

**1.** A uniform beam AB rests in contact with a smooth cylinder whose axis is horizontal, the vertical plane through the rod being perpendicular to the axis; the ends A and B of the rod are fastened by strings to the highest and lowest points of the cylinder, and the strings are horizontal. Find the tensions of the strings when the rod makes an angle $\alpha$ with the horizontal, and prove $\cos \alpha > \dfrac{\sqrt{5}-1}{2}$.

**2.** A circular table is supported symmetrically by three legs placed at the angles of an equilateral triangle ABC, of 30″ side. The table weighs 60 lbs., and a weight of 100 lbs. is placed on it at a point P, which is 4″ from BC and 8″ from AC. Find the load on each leg of the table.

**3.** A locomotive weighing 50 tons stands on a pair of rails on different levels, so that the inclination of the axles is 5°. The height of the C. of G. is 5′ 6″ and the distance between the centres of the rails is 5 ft. If there is no side pressure on the higher rail, find the side pressure on the lower rail, and also the normal pressure on the higher rail.

**4.** A uniform plank, weighing 200 lbs. and 20 ft. long, is placed with one end upon the ground, and rests upon a smooth fixed cylinder of 6 ft. diameter lying upon the ground. If the end upon the ground is 15 ft. from the line of contact of the cylinder with the ground, find the least horizontal force applied at the end upon the ground which will push the plank further up on the cylinder. $\mu$ between ground and plank $=0{\cdot}2$.

**5.** In the figure A and B are fixed points, 13″ apart, to which are hinged the stiff rods AD, 10″, and BF, 6″. The jointed parallelogram CDEF has its sides 5″ and $3\frac{1}{2}$″ long respectively, and is hinged to AD at the end D and the mid point C. It is also hinged to BF at F. When this system is in the position

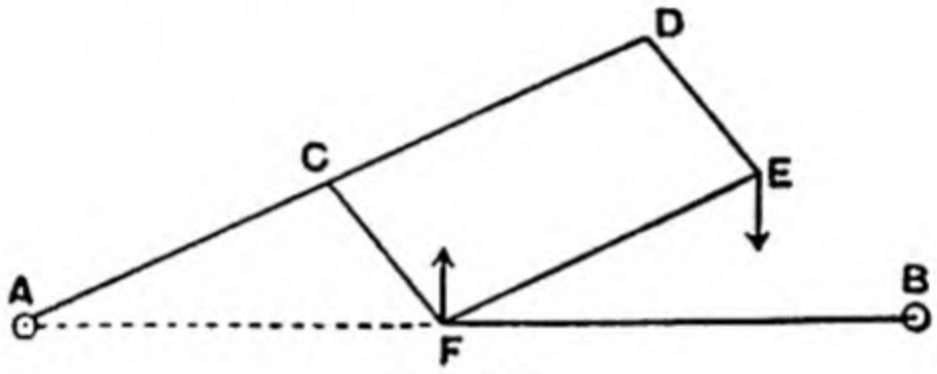



FIG. 287.

shown, F lying in the horizontal straight line AB, a weight of 1 lb. is hung from the joint E: this is balanced by a vertical force at F. Find the thrust in the rod CF, and hence, by a graphical construction, find the vertical force at F and the thrust in the rod FB.

## III.

**1.** A uniform circular table of wt. 45 lbs. is supported by 3 legs at points on the circumference A, B, C, the vertices of an equilateral triangle. A wt. of 15 lbs. is placed at the mid point of the line AB, a wt. of 20 lbs. at the mid point of the line AC and a wt. of 24 lbs. at the point midway between A and the centre of the table. Find the pressure on each of the legs of the table.

**2.** A photograph frame, 12″ high, is supported in an inclined position on a smooth table by a light support 8″ long, hinged to the back of the frame 8″ from the bottom. It is prevented from collapsing by a string 6″ long fastened to the frame and to the support 6″ from the hinge on each. Find the

pull in the string if the frame weighs 1 lb. and its C. of G. is halfway up. (C.S.C.)

**3.** A parallelogram ABCD is formed of four equal uniform rods freely jointed. It is hung from A, and the joints B and D are kept apart by a weightless rod. Show that the thrust of this rod is $2W \tan \theta$.
W = wt. of each rod ; $\theta$ the angle the rods make with the vertical.

**4.** If $\theta$ is the greatest slope on which a four-wheeled carriage can stand when both pairs of wheels are locked, find the greatest slopes on which the carriage can stand when

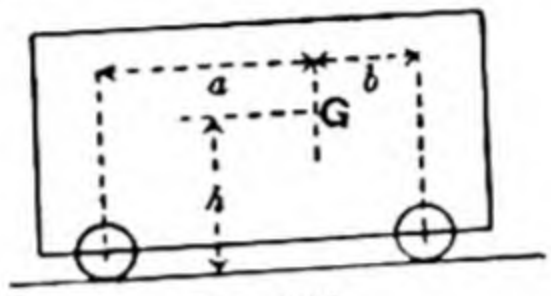



FIG. 288.

either the upper wheels or the lower wheels are locked, and show that the former slope will be greater than the latter if $a > b + h \tan \theta$.

**5.** The figure illustrates a method of raising a block of stone by means of light wedges C and D, inserted between fixed inclines A and B and the underside of the block. The coefficients of friction between the wedges and the block

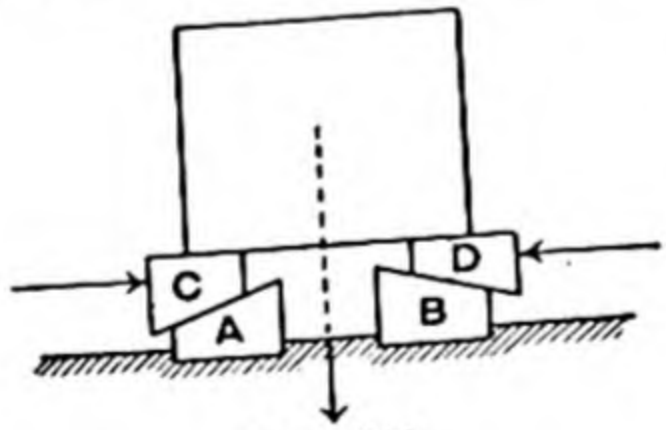



FIG. 289.

and inclines are $\tan \phi_1$ and $\tan \phi_2$ respectively, and the wedges are driven forward by equal horizontally applied forces. Prove that the efficiency of the arrangement for raising the load is $\dfrac{\tan \theta}{\tan \phi_1 + \tan(\theta + \phi_2)}$, where $\theta$ is the angle of one of the wedges. (C.U.)

## IV.

**1.** A weight of 1250 lbs. is suspended by three chains, each 15 ft. long, from 3 hooks in the ceiling, which form the corners of an equilateral triangle of side 18 ft. Find the pull in each chain.

**2.** A plane rectangular table ABCD is supported by 4 springs, all equal in length and in strength, placed one at each corner of the rectangle. If a load is placed on the table, show that, wherever it is placed, half of its wt. is supported by the springs at A and C, and the other half by the springs at B and D, the compression of the springs being proportional to the load they bear.

If AB = 6 ft., AD = 4 ft., and if the load be placed with its C. of G. at a distance of 2 ft. from both AB and AD, what is the thrust in each spring ?

**3.** Roof under wind pressure. Forces at C and D are inclined at 45° to the horizontal. The reaction at D is vertical, the direction of the reaction at A is determined from the force diagram. (Begin at B.)

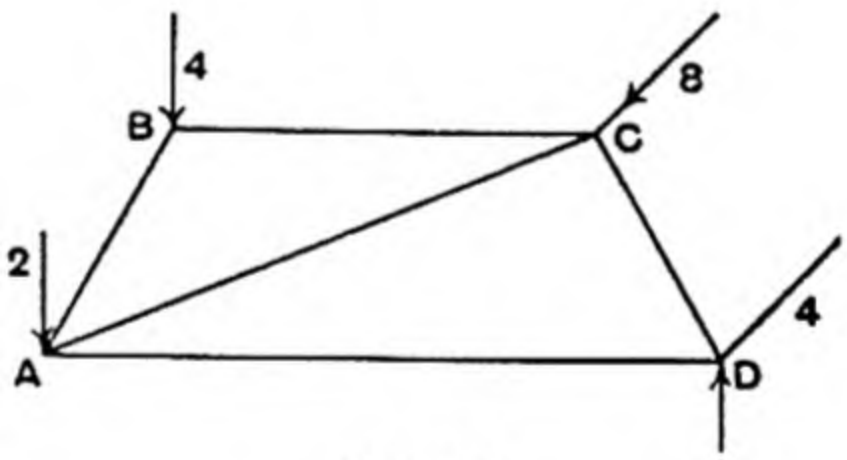



FIG. 290.

The framework is hinged to a wall at A and rests on a horizontal roller at D. Find the stresses in the bars.

$$BAD = ADC = 60°,\ AD = 30 \text{ ft.},\ BC = 20 \text{ ft.}$$

**4.** A sliding door moves on two wheels between smooth guides. When pushed by the horizontal force P, the near wheel refuses to turn and slides along the rail, while the other wheel rotates freely.

If the weight of the door is 600 lbs. and the coefficient

of friction for the wheel sliding on the rail is 0·2, find the magnitude of P required to move the door.

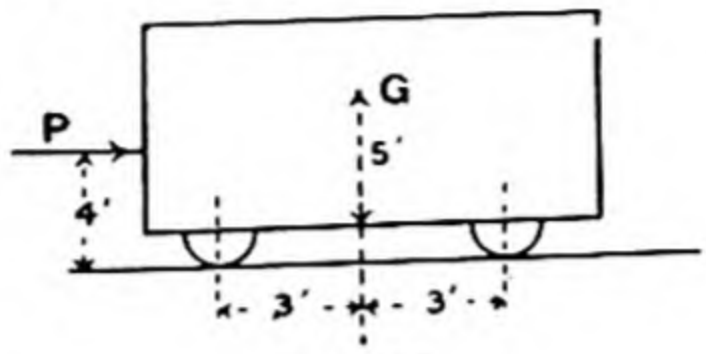



FIG. 291.

**5.** A uniform cylinder, length 1 yard, diameter 1 ft., mass 1 ton, the planes of whose ends are perpendicular to its axis, rests in limiting equilibrium with a point of its upper end against a smooth vertical wall, a point of its lower end against a rough horizontal plane and its axis inclined at 45° to the vertical. Find $\mu$.

A gradually-increasing horizontal force, acting towards the wall, acts on the cylinder at the point where it touches the plane. What is the magnitude of this force at the instant when the cylinder begins to slip ?

## V.

**1.** Three equal rods, each weighing 3 lbs. and 5 ft. long, are jointed together at one end so as to form a tripod. The middle points of the rods are joined by three light strings, each 1′ 6″ long: find the tension in each string when a weight of 10 lbs. is placed on the tripod, which stands on a smooth horizontal surface.

**2.** A light rod AC is arranged so that A may move freely in the horizontal line AD, and is controlled by another light

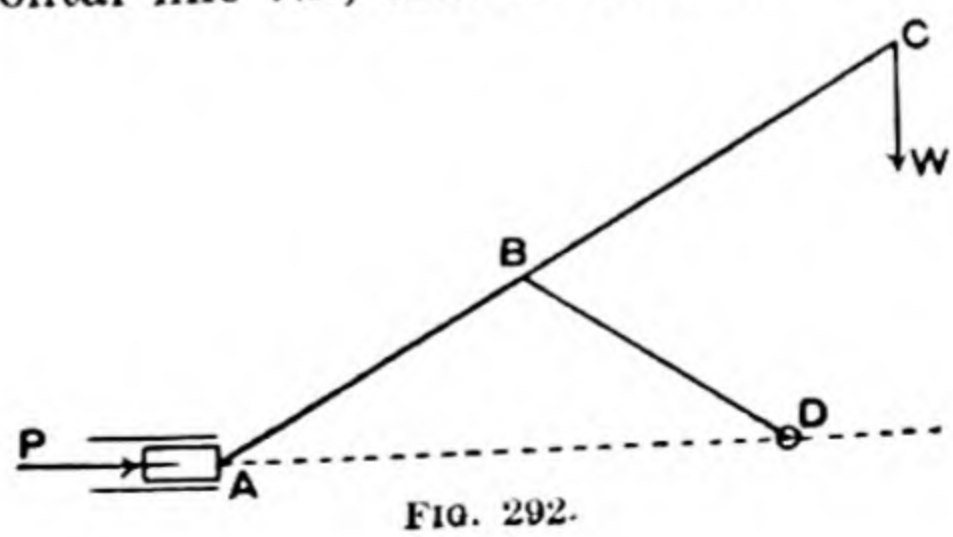



FIG. 292.

rod BD, hinged to AC at B, and capable of rotation about a fixed centre D. AB and BD are equal, and a weight W is hung at C. Neglect friction, and show that the horizontal force P, which must be applied at A in order to preserve equilibrium, is given by $P=\frac{W}{2}\left(1+\frac{BC}{AB}\right)\cot\theta$, where $\theta$ is the angle CAD. (C.S.C.)

**3.** Three bars AB, BC, CD hang from two fixed points A, D, on the same level, so that the system is in a vertical plane with BC horizontal, and the angles DAB, ADC equal to one another. The weights of AB and CD are W and W′ respectively, and BC is weightless. Prove that the system can be held in this position by a couple equal to $\frac{1}{4}(W-W')l$, $l$ being length of BC. Find the stress in BC.

**4.** A rigid framework in the shape of a rhombus, whose side is $a$ and acute angle $\alpha$, rests on a rough peg, the coefficient of friction being $\mu$. Prove that the distance between the two extreme positions which the peg can have in contact with a given side is $a\mu\sin\alpha$, provided $\mu<\tan\frac{\alpha}{2}$.

**5.** A hollow cylindrical vessel, 20 ft. in diameter, has three equal rectangular tanks, each 3 ft. wide and 4 ft. deep, running its full length, as shown in section. The vessel is floating, half immersed with tank No. 2 full and tanks 1 and 3 empty.

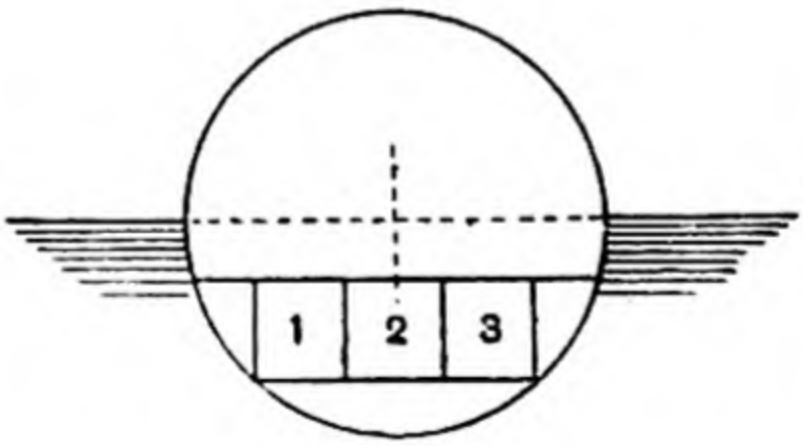



FIG. 293.

When the water in tank 2 is completely transferred to 3, the vessel is found to heel through an angle of 9½°. Show that, in its original state, the C. of G. is 1·37 ft. below its centre of figure. (C.U.)

## VI.

**1.** Three equal hemispheres of diameter 10 cms. are placed symmetrically with their centres 12 cms. apart on a rough horizontal table, and a sphere of equal diameter and double the weight is placed symmetrically, so as to be supported by the hemispheres. Assuming there is no friction between the sphere and the hemisphere, find the least coefficient of friction between the hemispheres and the table.

**2.** A six-wheeled railway truck is urged by a horizontal force of 3 tons against a log of wood fastened transversely to the railway line, as shown in the figure. If the wheels at

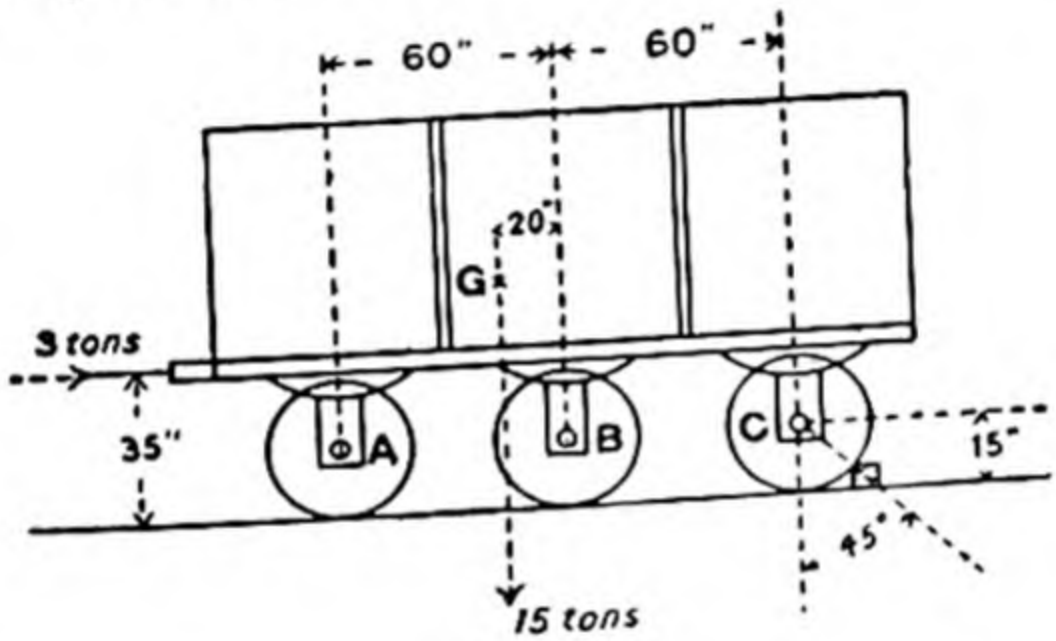



FIG. 294.

C are just lifted off the rails, determine the loads carried by A and B. (The wheels are supposed to rotate in their bearings without appreciable friction.) (C.U.)

**3.** Two equal smooth spheres of radius $r$ and weight W are placed inside a hollow cylinder of radius $a$, open at both ends, which rests on a horizontal plane. Prove that in order that the cylinder may not upset, its weight must be at least $2W\left(1-\frac{r}{a}\right)$.

**4.** ABCD is a quadrilateral formed by four uniform rods of equal weight but unequal lengths, loosely jointed together. If the system be in equilibrium in a vertical plane, with the rod AB supported in a horizontal position, prove

$$2\tan\theta = \tan\alpha - \tan\beta,$$

where $\alpha$, $\beta$ are the angles at A and B, and $\theta$ is the inclination of CD to the horizontal. Also find the stresses at C and D, and prove that their directions are inclined to the horizontal at angles $\tan^{-1}\left(\frac{\tan\beta - \tan\theta}{2}\right)$ and $\tan^{-1}\left(\frac{\tan\alpha + \tan\theta}{2}\right)$.

**5.** Plot a curve from the following coordinates:

| $x$ | 0·6 | 1·4 | 2 | 2·5 | 3·0 | 4·0 | 4·4 | 4·6 | 5·0 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0·8 | 1·0 | 1·2 | 1·4 | 1·65 | 2·4 | 2·8 | 3·1 | 4·0 |

Find the coordinates of the C. of G. of a uniform wire of this shape, the ends being (0·6, 0·8) (5·0, 4·0).

## VII.

**1.** In the wall crane CA is horizontal and BAC = 30°. Find the values of the tensions in AB (1) when the load is raised by the rope passing over the pulleys at A and D as shown, (2) when the rope passes only over the pulley A, and is led

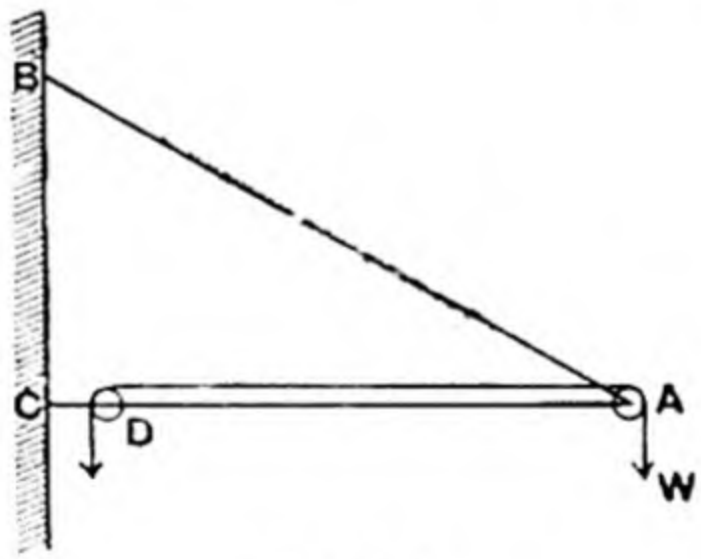



FIG. 295.

vertically downwards from it. The pulleys at A and D are carried on AC. Find the horizontal and vertical components of the force at the hinge C in the first case.

**2.** Four equal uniform rods are freely jointed at the ends so as to form a square ABCD. The square is hung up by the corner A, and is prevented from collapsing by a weightless

string joining the mid points of AB and BC. Show that the tension of the string is four times the weight of a rod.

**3.** The framework of weight $W_1$ is fitted with rollers at A and B so as to slide vertically in contact with a fixed rod of diameter $d$. A weight $W_2$ is placed at a distance $x$ from the

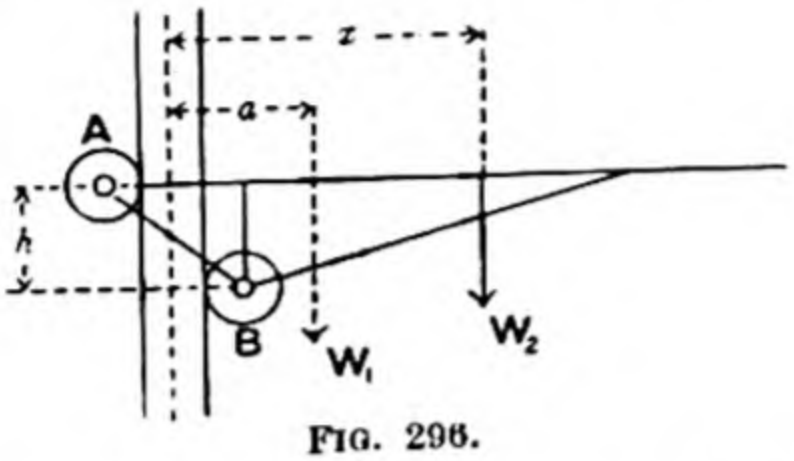



FIG. 296.

centre of the rod. If A is frictionless and B fixed, show that the framework will not descend unless

$$W_2 x < \left(\frac{d}{2} + \frac{h}{\mu}\right)(W_1 + W_2) - aW_1,$$

where $\mu$ is the coefficient of friction between the rod and the fixed roller. (C.U.)

**4.** Draw a regular hexagon ABCDEF of side 2 in. A force X of 2 lbs. acts along BA, and a force Y of 2 lbs. along EF. P acts along CD, Q along FD and R along EC. Find graphically the magnitudes of P, Q and R, if the system is in equilibrium.

Let the resultant of X and Y meet EC at F. Show that the resultant of P and Q acts along FD. Draw the force polygon for X, Y, R, and the resultant of P and Q. Hence find P and Q.

**5.** The figure represents a framework loosely jointed at its ends. The framework is supported by vertical reactions at

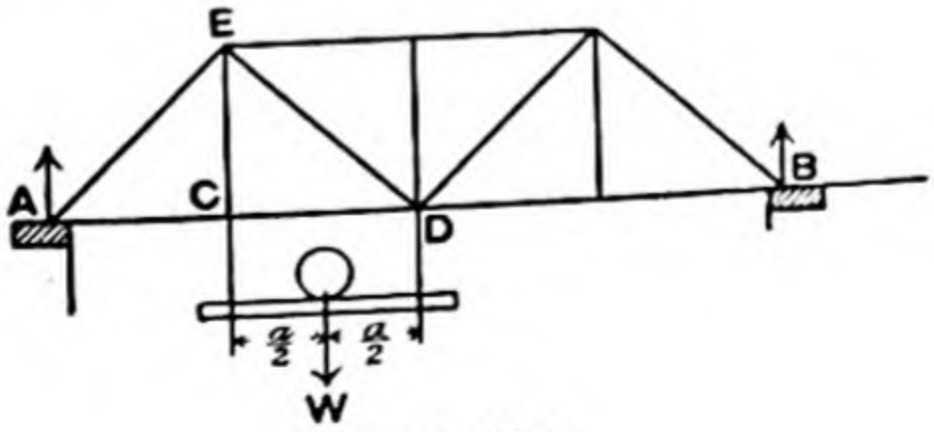



FIG. 297.

A and B, and carries a light platform suspended from C and D. A load W is placed at the centre of the platform. Sketch the force diagram for the left half of the framework. Show also that if W is shifted to a position on the platform distant $\frac{a}{3}$ from EC, the stress in the member DE is reduced to zero. All horizontal and vertical members are of length $a$. (C.U.)

## VIII.

**1.** A weightless string of length $5l$ has four equal weights W attached symmetrically to distances $l$ apart, and is in equilibrium with its ends fastened to two points in the same straight line. If $T_0$ is the tension of the lowest portion of the string, express in terms of $\frac{T_0}{W}$ the angles which the several portions of the string make with the horizontal.

**2.** Three equal rods AB, BC, CA, each of weight W and length $2a$, are jointed at B and C, and are in equilibrium with the two upper rods AB, AC in contact at A and resting on two smooth pegs, which are in the same horizontal line and distant $2c$ apart. A weight $w$ is hung from the mid point of BC; prove that $c > \frac{11}{12}a$ and $4w < \frac{12c - 11a}{a - c}$. W.

**3.** Three smooth equal spheres of radius 1 ft. and weight 1 lb. are placed inside a smooth hemispherical bowl of radius 4 ft. Find the pressure between the bowl and the spheres.

**4.** A ring C, at the end of a uniform heavy bar BC of length $5\frac{1}{3}$ ft., slides on a vertical rod AD. A string 6 ft. long is fastened to the end B of the bar and to a point A on the rod 1 ft. above C. If the ring is on the point of slipping upwards in the given position, find graphically the position of BC when C is on the point of slipping downwards.

**5.** Two equal friction wheels of radii R in. turn on axles B and C of radii $r$ in. A wheel A of weight W is attached to an axle of radius $a$ in., which rests on the circumferences of the

first two wheels as in the figure. Neglecting the weights of the wheels B and C, show that the least couple which will rotate

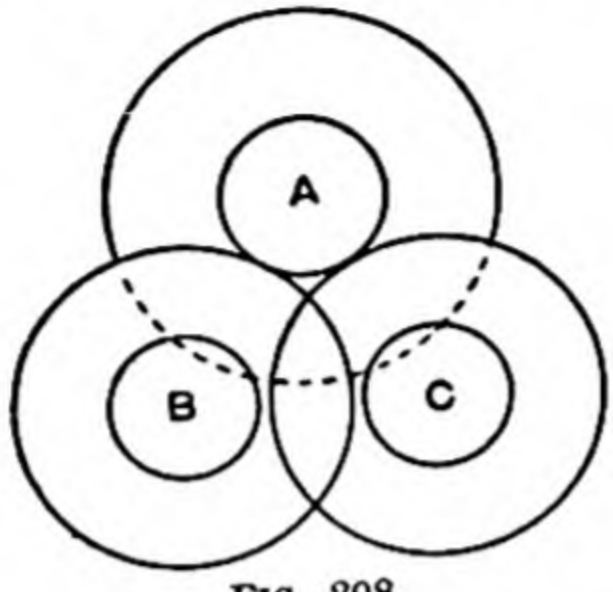



FIG 298.

the wheel A is $\dfrac{Wa \sin 2\theta}{\sqrt{3}}$, where $\sin\theta = \dfrac{r}{R}\sin\phi$, the centres of the three wheels forming an equilateral triangle and $\phi$ being the angle of friction between the wheels and the axles. (C.U.)

# APPENDIX.

THE following examples, which involve a knowledge of Trigonometrical manipulation, are inserted to meet the needs of students who require such examples for examination purposes.

**1.** A uniform rectangular plate ABCD of weight W can turn in a vertical plane about a smooth pivot at the mid-point of its upper edge AB. It is tilted through an angle $\theta$ by an outward horizontal pull P at D. Prove that $\tan\theta = \frac{2bP}{bW + aP}$ where $AB = 2a$, $BC = 2b$.

**2.** A uniform rod AB of length $2a$ rests in a vertical plane with its upper end A against a smooth wall, the other end being kept at a distance $2c$ from the wall by a string fastened to B and to a nail in the wall. Prove that the thrust against the wall is $\frac{Wc}{2\sqrt{a^2 - c^2}}$, W being the weight of the rod.

**3.** A horizontal rod has its ends on two inclined planes of inclination $\alpha$ and $\beta$, and lies in a vertical plane at right angles to the line of intersection of the planes. Prove that the C. of G. of the rod divides it into two parts in the ratio $\tan\alpha : \tan\beta$.

**4.** Two weights $W_1$ and $W_2$ are connected by a string which is placed over the surface of a smooth cylinder whose axis is horizontal. The arc subtended at the centre by the string is $\alpha$ ($\alpha < 2\pi$). If the radius joining $W_1$ to the centre makes an angle $\theta$ with the vertical, prove that $\tan\theta = \frac{W_2 \sin\alpha}{W_1 + W_2 \cos\alpha}$.

**5.** A uniform bar AB of weight W and length $2a$ can turn about a smooth pivot at its upper end A, and the lower end B is kept at a distance $2b$ from the vertical by a horizontal force applied at B. Prove that the reaction at the hinge is $\frac{W}{2}\sqrt{\frac{4a^2 - 3b^2}{a^2 - b^2}}$.

**6.** A uniform rectangular plate ABCD is free to turn in a vertical plane about a hinge at A, and the plate is held in a position in which D is above and B is below the level of A by a horizontal force equal to the weight of the plate applied at a point P of AB. Prove $AP = \frac{1}{2}(AD + AB \tan \alpha)$ where $\alpha$ is the inclination of AD to the horizontal.

**7.** The sum of the weight of a certain Roman steelyard and of its movable weight is S; the fulcrum is at a point C and the body to be weighed is hung at the end B. The steelyard is graduated, and after graduation the fulcrum is shifted towards B to another point C′. A body is then weighed, the old graduations being used, and the apparent weight is W. Prove that the true weight is greater than the apparent weight by $\frac{(S+W)\,CC'}{BC'}$.

**8.** A two-wheeled carriage of weight W has to be drawn over an obstacle of height $h$ which lies across the road, by means of a pull P applied horizontally at the level of the axles. Prove that the least value of P is $\frac{(2ah - h^2)^{\frac{1}{2}}\,W}{a-h}$ where $a$ is the radius of the wheels.

**9.** A string is tied tightly round two smooth cylinders of radii $r_1$ and $r_2$. If T is the tension of the string, prove that the pressure between the cylinders is $\frac{4T\sqrt{r_1 r_2}}{r_1 + r_2}$.

**10.** Two smooth inclined planes slope in opposite directions making angles $\alpha$ and $\beta$ with the horizon. A cylinder at weight W is placed between them with its axis parallel to their line of intersection. Prove that the pressures on the planes are $\frac{W \sin \alpha}{\sin(\alpha+\beta)}$ and $\frac{W \sin \beta}{\sin(\alpha+\beta)}$.

**11.** Two uniform boards of weight W, W′ rest with their upper ends in contact with two parallel smooth walls, their lower ends being in contact on a smooth horizontal floor. Prove that if the boards are in equilibrium when they make angles of $\theta$ and $\theta'$ with the floor respectively then

$$W \cot \theta = W \cot \theta'.$$

**12.** If two forces acting on a particle be represented by $m$ times the line OA and $n$ times the line OB respectively, their resultant is represented by $m+n$ times the line OG where G divides AB so that $m\mathrm{AG}=n\mathrm{BG}$.

**13.** ABC, A′B′C′ are any two triangles, G, G′ their C. of G. Show that forces at a point parallel and proportional to AA′, BB′, CC′, 3G′G will be in equilibrium.

**14.** A vertical post is held upright in contact with the ground by two cords attached to pegs on the ground at equal distances $c$ from the foot of the post, and attached to the post at heights $a$ and $b$ above the ground, the cords being on either side of the post in a vertical plane. Prove that the tangent of the angle which the reaction of the ground makes with the vertical cannot exceed $\frac{\frac{1}{2}c(a-b)}{ab}$.

**15.** A uniform rod rests over a fixed smooth sphere with its lower end pressing against a smooth vertical plane which touches the sphere. If $\theta$ is the angle the rod makes with the vertical when in equilibrium, prove that $a=2l\sin\frac{\theta}{2}\cos^3\frac{\theta}{2}$ where $l$ is the length of the rod, $a$ the radius of the sphere.

**16.** A heavy triangle ABC is hung up by the angle A, and the opposite side is inclined at an angle $\alpha$ to the horizon. Show that $2\tan\alpha=\cos\mathrm{B}\sim\cot\mathrm{C}$.

**17.** ABCD is a parallelogram, and forces are completely represented by the lines AB, BC, CD, DA, CA, DB. Prove that their resultant is a force of magnitude 2CB, and direction parallel to CB.

**18.** ABCD is a quadrilateral having the angles at A and D right angles and CB = CD. Forces P, Q, R acting along AB, CA, and AD respectively are in equilibrium. If $\mathrm{P}=\frac{2}{\sqrt{3}}\mathrm{R}$, prove that $\mathrm{AB}=\frac{1}{3}\mathrm{BC}$ or $\frac{3}{5}\mathrm{BC}$.

**19.** One end of a heavy uniform rod AB can slide along a rough horizontal rod AC to which it is attached by a ring. B and C are joined by a string. If ABC is a right angle when

the rod is on the point of sliding, $\alpha$ the angle AB makes with the vertical, prove $\mu = \dfrac{\tan \alpha}{2 + \tan^2 \alpha}$.

**20.** A body of weight W can just be moved along a rough horizontal plane by a force P acting at an angle $\theta$ to the plane. Prove $P = W \sin \lambda \sec (\theta - \lambda)$, and hence show that P is least when $\theta = \lambda$.

**21.** A uniform ladder of length $l$ rests on rough horizontal ground with its upper end projecting slightly over a smooth horizontal rail at a height $a$. If the ladder is about to slip and $\lambda$ is the angle of friction with the ground, prove

$$\tan \lambda = \frac{a\sqrt{l^2 - a^2}}{l^2 + a^2}.$$

**22.** Two equal rough thin horizontal and parallel bars A and B are at a distance $2a$ apart in a line inclined at an angle $\theta$ to the vertical. A rod passes over bar A and under bar B and is just kept from sliding down by friction. Prove that the c. of G. of the rod must be at a distance from the upper bar equal to $\dfrac{a(\cot \theta - 1)}{\mu}$.

**23.** At the foot of a rough plane inclined at $\alpha$ to the horizontal is a vertical smooth wall. A uniform beam of weight W is placed in a plane at right angles to the wall with its upper end leaning against the wall and its lower end resting on the plane. Prove that when the beam is about to slip down the wall its inclination $\theta$ to the horizontal is given by $2 \tan \theta = \cot (\alpha + \lambda)$.

**24.** A ladder of length $l$ rests against a rough vertical wall in a plane at right angles to it. If the coefficient of friction is $\mu$ both at the ground and at the wall, prove that when a man ascends the ladder the greatest distance of the c. of G. of man and ladder from the foot is $\dfrac{\mu l(\mu + \tan \alpha)}{1 + \mu^2}$ when $\alpha$ is the inclination of the ladder to the ground.

**25.** A uniform triangular lamina ABC of which the angle BAC $= 90°$ is suspended by a string of length AB, attached at A and to a point in a vertical wall so that the triangle hangs in a vertical plane perpendicular to the wall with the point B

resting against the wall. Prove that in the position of equilibrium the inclination of the string to the wall is $\tan^{-1}(\frac{1}{4}\tan B)$.

**26.** An equilateral triangular lamina ABC rests in equilibrium in a vertical plane with AB, AC in contact with two smooth pegs in the same horizontal line at a distance $a$ apart. Prove that if AD, the perpendicular from A on BC, is not vertical it must make an angle $\theta$ with the vertical where $\cos\theta = \frac{h\sqrt{3}}{6a}$ and $h = AD$.

**27.** Two equal rods AB, AC freely jointed at A have their ends B, C connected by a light string. B and C rest symmetrically on two smooth planes inclined at angles $\alpha$ to the horizontal, with their slopes facing one another, the plane of the rods being vertical. Show that the tension in the string is $\frac{W}{2}(\tan\theta - 2\tan\alpha)$; W being the weight of a rod and $2\theta = BAC$.

**28.** ABC is a triangle, right angled at C, formed of three uniform rods jointed at their ends. It is suspended by a string attached to the mid-point of AB. Prove that the reaction at C is $\frac{1}{2\sqrt{2}}cw$, where $w$ is the weight per unit length of each rod.

**29.** A thin straight uniform rod of length $2c$ and weight $w$ rests over the edge of a smooth thin hemispherical bowl of radius $a$ and weight $w'$ so as to make an angle of 30° with the plane of its rim. The bowl rests on a horizontal plane; prove that $\frac{a}{c} + \frac{w}{w'} = \frac{\sqrt{3}}{2}$.

**30.** A heavy uniform rough rod slides through a light ring attached by a string to the edge of a table, and remains in limiting equilibrium with one end pressing on the under surface of the table. If $2a$ be the length of the rod, $x$ the distance of its point of contact with the table from the edge, $\alpha$ the inclination of the rod to the table, $\lambda$ and $\lambda'$ the angles of friction at the table and the ring, prove

$$\frac{\tan(\alpha - \lambda')}{\tan\lambda} = \frac{a\cos\alpha - x}{a\cos\alpha}.$$

# ANSWERS.

## CHAPTER I.

### Examples I. (p. 15.)

**1.** $1\frac{1}{4}$ ft. from mid point. **2.** 3·6 ft. from axis, 316 lbs. **3.** $3\frac{1}{7}$ lbs. **4.** 20 grams. **5.** 60·6 grams. **6.** 140 lbs. **7.** 19·9 grams. **8.** $1\frac{1}{4}$ ft. from centre. **9.** $4\frac{1}{8}$ ft. **10.** $\sqrt{W_1 W_2}$.

### Examples II. (p. 22.)

**1.** 20 lbs.-ft., 80, 60. **2.** $\frac{1}{4}$ lb. **3.** 1·4 ft. from B. **4.** 75 grams. **5.** 2 ft. **6.** 15 lbs. **7.** 14·3 stone. **8.** 3 ft. **9.** 20 lbs., 0 and 35 lbs., 50 lbs., 70 lbs. and 0. **10.** 34 lbs. **11.** 2·46 ft. from A. **12.** $6\frac{9}{7}$ ft., 7 lbs. **13.** Man nearer the bottom by $11\frac{3}{7}$ lbs. **14.** 91·3 lbs. **15.** $183\frac{1}{3}$, $116\frac{2}{3}$ lbs. ; $303\frac{1}{3}$, $236\frac{2}{3}$ lbs. **16.** 1 ft. **17.** 3·75 ft. from end. **18.** 20 lbs. to end 4 ft. from prop. **19.** $P = 360 - 20x$. $Q = 20x + 120$. $P = 0$ when $x = 18$ ft. **20.** $43\frac{1}{3}$, $66\frac{2}{3}$ lbs., 10 lbs. on the man. **21.** (i) 2 in. from B ; (ii) 1 in. ; (iii) 22 lbs. **22.** (i) at G ; (ii) $6\frac{9}{7}$ in., 12 in., 16 in. ; (iii) 10 lbs. **23.** $b$ lbs. where $b$ in. = AB (Fig 24). **24.** 22 lbs. **25.** $56\frac{1}{2}$ lbs.

### Examples III. (p. 30.)

**1.** 14·14 lbs. **2.** 27·5 lbs. **3.** 2 lbs. **4.** 6 lbs. **5.** 6 lbs. **6.** 192 grams. **7.** $7\frac{1}{2}$ lbs., 6 lbs. **8.** 42 grams or 120 grams. **9.** 3·4 lbs. **10.** 70° 32′. **11.** $\frac{2}{7}$ ton. **12.** ·031 ton. **13.** 5·4 lbs. **14.** ·18 lb. **15.** 26° 36′. **17.** ·9 cwt. **18.** 960 lbs. **19.** $\frac{8\sqrt{3}}{9}$. **20.** 832 grams.

## Examples IV. (p. 36.)

**1.** 11 lbs., $\frac{4}{11}$ of diag. from 3 lbs.; 5 lbs., $\frac{4}{5}$ of diag. from 3 lbs.
**2.** (i) 10, 90 cms. from 100; (ii) 1, 990 cms. from 100; (iii) no equilibrant.
**4.** 180 lbs.-ft. **6.** Draw BD in the opposite direction to AC.
**7.** 6 lbs.

# CHAPTER II.

## Examples V. (p. 43.)

**1.** 500 ft.-tons, 1,120,000 ft.-lbs. **2.** 50,000 cm.-grams.
**3.** 141·4 ft.-tons. **4.** 20 ft.-tons, $33\frac{1}{3}$ ft.-tons, $13\frac{1}{3}$ ft.-tons.
**5.** 15 lbs. **6.** 600 ft.-lbs. **7.** $\frac{2}{3}$ oz. **8.** 87·5 %
**9.** 72 lbs. **10.** 1·4 ft.

## Examples VI. (p. 51.)

**1.** $83\frac{1}{3}$ %, $\frac{2}{3}$. **2.** 80 %. **3.** $7\frac{1}{2}$ in. **4.** 2·4 cwt.
**5.** 56 %. **6.** 157 lbs., 89 %.
**7.** 68, 83, 89·5, 91·6, 92 %, 1·4 lbs. **8.** P = ·32W + 88; 37·2 %.
**9.** $(\frac{7}{8})^6$P, $(\frac{8}{7})^6$P. **10.** 69 lbs., 30 lbs.
**11.** 21·5 lbs., 75 lbs.

*Archimedes' System.* (p. 54.)

$e$ = 24·6 %, 36·2 %, 41·6 %, 48·4 %, 51·3 %.
Friction, 6·6, 7·4, 9·4, 10·9, 13·6.

## Examples VII. (p. 54.)

**1.** 100 lbs., 250 lbs. **2.** 9·7, 26·3, 72·3 %, 87·5 %.
**3.** 127·3 lbs. **4.** 2, 18·5 %, 51·3 %, P = ·74W + 19·7.
**5.** 4, 50, 56 %. **6.** 35 %, 54 %. **7.** $T = \frac{W}{5}$.
**8.** 53, 61, 63, 65, 66 %; P = ·14W + 25, 13 grams.

## Examples VIII. (p. 60.)

**1.** 216 lbs. **2.** 7·6. **3.** 47 % nearly.
**4.** 4000 lbs. **5.** $6\frac{2}{3}$ lbs. **6.** 31·1 lbs.
**7.** 655 lbs. **8.** 1·3 metres. **9.** $5\frac{3}{4}$ in.

## Examples IX. (p. 64.)

**1.** 24·8 %.
**2.** 44 lbs.
**3.** 804 lbs., 482 lbs.
**4.** 35 lbs.
**5.** 140 lbs.
**6.** 27 lbs./sq in.
**7.** $W = -23 + 10{\cdot}3P$.
**8.** 1·5 in.
**9.** For equilibrium $n = 8{\cdot}3$.
**10.** 2261 lbs.
**11.** 84 lbs.
**12.** $\frac{1}{10}$. **13.** $\frac{8}{52}$, 81.

# CHAPTER III.

## Examples X. (p. 75.)

**1.** 38 lbs., 28 lbs.
**2.** 23·3 lbs.
**3.** 22·9 lbs.
**4.** 36° 42′ ; 48° 22′.
**5.** 80 lbs.
**6.** 13·1, 10·6, 10, 10·6, 13·1 lbs. Perpendicular to AB.
**7.** $1\frac{1}{2}$ tons, $1\frac{1}{4}$ tons.
**8.** 18·8 lbs., 6·8 lbs.
**9.** 1·1 cwt., 1·0 cwt.
**10.** 1·2 cwt.
**11.** $\frac{3}{5}$ cwt.
**12.** 554 grams.
**13.** 62 5 lbs., 37·5 lbs.
**14.** 65 5 lbs. wt.
**15.** ·42 cwt.
**16.** 15·7 lbs.
**17.** 7·7 cwt. at 36·8° to the vertical.

## Examples XI. (p. 80.)

**1.** 11·2 lbs.
**2.** 17·4 lbs., 10·6 lbs.
**3.** 3·9 lbs.
**4.** 4·62 lbs.
**5.** 1·85 lbs., 1·85 lbs.
**6.** AC
**7.** 2·24 lbs., 27° to AD.
**8.** 6·8 lbs., 5·3 lbs.
**9.** 114·6 lbs.
**10.** 169°.

## Examples XII. (p. 82.)

**1.** 3·76 lbs.
**2.** 15 lbs.
**3.** 28·3 lbs.
**4.** 53 lbs.
**5.** 28·9 lbs.
**6.** 12 3 lbs.
**7.** 61·3 lbs.
**8.** ·745 lb.
**9.** 1 49 lbs.
**10.** 3·42 lbs., 9·40 lbs., 3·42 lbs.

## Examples XIII. (p. 88.)

**1.** 120°.
**3.** 1·29 lbs.
**4.** 64·3 lbs
**5.** 400 ft.-lbs , 13·68 lbs , 29·24 ft., 400 ft.-lbs
**6.** 689 ft.-lbs.
**7.** 0·36 ton.
**8.** 3648 lbs. wt.
**9.** 47·8 ft.-tons.
**10.** 9·7 lbs., 341 lbs.-ft.
**11.** 18 8 lbs.
**12.** 7·82 lbs.
**13.** 7·07 lbs., 45° to horizontal.
**14.** 8·66 lbs.
**16.** ·031 ton.

## Examples XIV. (p. 93.)

**1.** 41·95 lbs. **2.** 105 lbs., 53·2 lbs. **3.** 1500 grams.
**4.** 520 lbs. **5.** 2·13 lbs. **6.** 11·4 lbs., 7·4 lbs.
**7.** 28·8 lbs. **8.** 4·6 lbs. **10.** 10·6 lbs., 13·7 lbs.

## Examples XV. (p. 98.)

**1.** (i) P vertically below A; (ii) Strings make equal angles with the vertical; P vertically below A.
**2.** 8·4 lbs., 6·0 lbs., 33° 42′ to horizontal.
**3.** 5·4 lbs., 22° 46′ to vertical. **4.** 3·1 ft.
**5** 2 88 lbs., 5·77 lbs., 30° to vertical.
**6.** 84·3 lbs., 71° 30′ to the ground. **7.** 2·4 lbs., 4 lbs.
**8.** 3·97 ft., 2·6 lbs.
**9.** 3·9 lbs. at 32° 30′ to vertical, or 5·3 lbs. at 6° 20′ to horizontal.
**10.** 92 lbs., 106 lbs. **11.** 1·14 ft. from lower end.
**13.** 1·2 ft. from A, 35 lbs., 45 lbs. **14.** 3·5 lbs., 2·9 lbs
**15.** 27·8 lbs., 12·8 lbs.
**16.** String perpendicular to rod; 2·5 oz.; 36° 45′ to vertical.
**17.** 16°, 1·4 lbs, 4 2 lbs.
**18.** At A 3·4 lbs., at 83° with BA; at B and C, 1·97 lbs. along BC.
**21.** 5 lbs., 10 lbs. **22.** 15 lbs.

## Examples XVI. (p. 106.)

**1.** 23·3 lbs. **2.** 29·9 lbs., 58° 48′ with AO.
**3.** 220 grams, 82 grams, 188 grams. **4.** 11·3; at 45° to axes S.W.
**5.** 24 2 lbs., 12° 30′. **6.** 30·9; 26 lbs. N.E.; 43 lbs. S.
**7.** 7·25 lbs., 3·2 lbs.

## Miscellaneous Examples.

### A. (p. 108.)

**1.** 11 oz., 4 oz. **2.** 20 lbs., 10 lbs. **3.** 53° 8′.
**4.** 1590 lbs. **5.** 73 lbs.

### B. (p. 108.)

**1.** 1 lb. lost. **2.** Resultant is along AD. **3.** 35 lbs.
**4.** 2·12 in. **5.** 45·7 %.

## C. (p. 109.)

**1.** 2 lbs., 1·73 lbs. **2.** 7·85 in. **3.** 4·47 at 63° 24′ with OX.
**4.** 24, 45 %. **5.** 7½ cwt.

## D. (p. 109.)

**1.** 4·8 lbs., 7·2 lbs., 1·7 lbs. **2.** AD horizontal. Vertically.
**3.** $1:\sqrt{2}$. **4.** 240, 20·7 lbs. **5.** 12·8 lbs.

## E. (p. 110.)

**1.** 19·2 in. **2.** 6 lbs., 13 lbs., 12 lbs. **3.** ·53 ft.
**4.** 4·1 lbs., 4·5 lbs. **5.** 1·05W, 18° 26′.

## F. (p. 111.)

**1.** 2·6 lbs., 141°. **2.** 75 lbs., 125 lbs. **3.** 25 lbs.
**4.** 11·1, 23·5, 270 approx. **5.** ·74 lbs., 1·3 lbs.

## G. (p. 112.)

**1.** 2 lbs., 4 lbs. **2.** 24 lbs., 102 lbs. **3.** 2·18 lbs. **4.** 2·01 kilos.
**5.** The two reactions meet on the circumference of the vertical section of the bowl.

# CHAPTER IV.

## Examples XVII. (p. 118.)

**2.** 10 lbs. **3.** ·25. **4.** 5 lbs. **5.** 8·45 lbs.
**6.** 47 lbs., ·36. **7.** 26·6 lbs., 9·4 lbs. **8** ·49, 55 lbs., ·49.
**9.** Vertical 30°. **10.** 32·14 grams, 38·3 grams, ·23.
**11.** 4·4 lbs., 7·84 lbs., ·47. **12.** 25 lbs. **13.** 14·4 lbs., ·68 lb.
**14.** 12·2 lbs. ($\mu = \cdot 31$). **15.** 5050 ft.-lb.

## Examples XVIII. (p. 123.)

**1.** ·84. **2.** 23° 30′. **3.** ·27. **4.** 16° 42′.
**5.** 15° to horizontal, 29 lbs. **6.** 8·4 in. from mid point, 27°.
**7.** 30° + λ, 85·8 lbs. **8.** 30°, 14 lbs. **9.** 0·4. **10.** 0·15
**11.** 0·19, 0·95 lb. **12.** 0·17. **13.** ·037. **15.** 0·52.

# CHAPTER V.

## Examples XIX. (p. 133.)

**1.** $\frac{5}{9}$ ft. **3.** 33° 42′. **5.** $4\frac{5}{8}$ cms. from top edge.
**6.** 2·6 cms. from AB, 2·4 cms. from CD. **7.** At c. of g.
**8.** 1 in. from common base. **9.** $\bar{x}=0{\cdot}6$, $\bar{y}=2\ 6$.
**10.** If the line joining A to the c. of g. cuts CD, then BC will leave the plane.

## Examples XX. (p. 140.)

**1.** $2_{[}\frac{1}{2}$ ft. from 1 lb. **2.** $\frac{1}{2}$ ft. from AD, $\frac{11}{16}$ ft. from AB.
**3.** $13\frac{1}{3}$ in from heavier end. **4.** 22·4 in.
**5.** 8·6 in. from edge of the wood. **6.** 4·2 in
**7.** 6″ from line (2, 6) ; $8_{17}^{3}$″ from line (2, 3).
**8.** 0·84 ft. **9.** $3\frac{3}{8}$ in.

## Examples XXI. (p. 143.)

**1.** $\frac{3}{2}$ in., $\frac{5}{3}$ in. **2.** 1·3$a$ ($2a$ = side of square). **3.** $4_{11}^{5}$ in.
**4.** $_{1}\frac{1}{2}$ of the diagonal from the centre. **5.** $\frac{1}{8}$ ft.
**6.** $\frac{3}{5}$ in. **7.** $1\frac{1}{8}$ in. **8.** 1·8 in.
**9.** 7 4 ft. from bottom. **11.** 0·25 in. **12.** 11°.
**13.** $\frac{1}{8}\frac{1}{8}$ of the altitude. **14.** $\frac{1}{3}\frac{1}{3}$ of the altitude. **15.** 3·098 in.
**16.** AD would leave the plane. The moment of the wt. of ABC about AC is > the moment of ACD.

## Examples XXII. (p. 149.)

**1.** 45°. **2.** 9·53 in. nearly. **3.** 4·58 in.
**4.** Upper, $\frac{1}{2}$ its length over lower. Lower, $\frac{1}{4}$ over the edge. Lowest, $\frac{1}{8}$ of its length over the edge.
**5.** Four. **6.** ·873 ft. **7.** W.
**8.** 1 lb. **10.** 11·5 lbs., 10·7 lbs.

## Miscellaneous Examples.

### H. (p. 151.)

**1.** 1·36 lbs. vertical. **2.** W opposite W, 2W opposite 2W.
**3.** 4 m. **4.** 1·78 lbs., 0·5 lb. **5.** 29° 19′.

## I. (p. 151.)

1. $\frac{w}{3}, \frac{2w}{3}$.
2. Six.
3. $P = 0{\cdot}11W + 4{\cdot}8$; 62, 73·5, 78, 89, 90 %.
4. 11·34 in. from left edge.
5. 0·65.

## J. (p. 152.)

1. 1 lb.
2. 2·04 lbs.
4. 0·35 ft.

## K. (p. 153.)

1. AB makes 14° with vertical.
2. $2d \cot \alpha$, $\frac{d \cos \alpha}{\sin(\alpha+\beta)}$, where $2\beta$ = angle of cone.
3. $\mu = 0{\cdot}58$ for slipping.
4. $\tan \lambda = {\cdot}615$; $\therefore$ it slips.
5. 37·9, 46·2, 46·2 lbs.

## L. (p. 154.)

1. 14°.
2. 7·1 lbs.
3. 11·2 lbs. from B to mid point of DC.
4. 16·1 lbs., 27°.

## M. (p. 155.)

1. 41·2 grams.
2. 75 grams, vertical.
3. 2·1 lbs., 4·53 lbs.
4. 2·67 ft.
5. 20° 6′.

## N. (p. 156.)

1. Upper, 70 lbs. Lower, 92 lbs. at 43° 36′ to horizontal.
2. ·97W, ·26W.
3. $P = \frac{l(R_2 - R_1)}{x}$, $W + P = \frac{(R_1 - R_3)l + R_3 y)}{y}$.

## Revision Questions. (p. 158.)

1. $4\frac{1}{8}$ lb.
2. $\frac{3}{8}$″ from mid point of AB., 6 lbs.
3. 4, $66\frac{2}{3}$%, 50 lbs.
4. 1·35 lb.
5. 120 lbs.
6. Falls back, ·47 cwt.
7. 42 lbs., 30·1 lbs., 70° 19′ to the slope. Yes.
8. 67 lbs. Tension.
9. 1.24 ton.
10. 1·73 ft.
11. 1·44 lb.
12. 3·5 lbs, 5·6 lbs., $2\frac{1}{2}$ ft./lbs. each, unaltered.
13. 1·15 lb., ·29 lb.
14. 30·8 lbs.
15. 1·5 lb.
16. 3″ from centre.
17. ·64 ft. below centre.
18. $\bar{x} = 4''$, $\bar{y} = 0{\cdot}6''$ (from D); 3 lbs. $\tan^{-1}\frac{3}{8}$ to vertical.
19. $\bar{x} = 10{\cdot}2$, $\bar{y} = 3{\cdot}76$ (from D).

# ANSWERS.

## CHAPTER VI.

### Examples XXIII. (p. 163.)

**1.** (*a*) $\frac{8}{9}$" from 10 lbs.; (*b*) $\frac{7}{8}$" from 12 lbs.; (*c*) $1\frac{2}{13}$" from 4 lbs.; (*d*) $\frac{4}{5}$" from 8 lbs.

**2.** $\frac{1}{4}$ ton, $\frac{3}{4}$ ton.

**3.** 13·7 cwt., 23·3 cwt.

**4.** 6·25 cwt., 8·75 cwt.

**5.** 1 ton, 12 tons.

**6.** $8\frac{3}{4}$ cwt., 6·25 cwt.

**7.** 23 tons, 22 tons.

**8.** 1·7 and 11·7.

**9.** 10 tons.

**10.** 4·7" from mid point.

**11.** 6 ft. from A.

**11.** 3·2 ft. from 20 lbs., 1·6 ft. from 20 lbs.

### Examples XXIV. (p. 169.)

**1.** Cuts AB $1\frac{1}{7}$ ft. from A, BC $1\frac{1}{11}$ ft. from B, AC $5\frac{1}{3}$ ft. from A.

**2.** At 45° to 2 lbs. and 6 lbs., $2\frac{2}{3}$ ft. from that corner.

**3.** 2 lbs. along DC.

**4.** Cuts AB produced 3·8 ft. from B, BC produced 9·7 ft. from B, AC produced 5·4 ft. from A. R = 2·1 lbs.

**5.** Cuts AC 3·1 cms. from A, AB 2·7 cms. from A, BC produced 10 cms. from B.

**6.** 11·2 lbs.

**7.** P = 12·36, Q = 15·5, R = 5·6.

**8.** 4 lbs. Force at pin = 16·5 lbs. cuts DC at E, where $DE = \frac{DC}{4}$.

**9.** 15 lbs.

## CHAPTER VII.

### Examples XXV. (p. 179.)

**1.** AB = +2·83, AC = +3·78, BC = −2·12.

**2.** AB = AC = +6·73, BC = −6·25.

**3.** AB = AC = +5·4, BC = −5·0.

4. BE = CF = +16·2, AE = AF = +10·8, ED = DF = +5·4,
   BD = DC = −15·0, AD = −4·0.
5. BE = +20·9, BD = −19·4, EA = +14·3 = AF, ED = +6·6,
   AD = −4·5, DF = +5·3, FC = +19·5, DC = −18·1.
6. AB = AC = +11·2, BD = DC = −10·1, AD = −2.
7. BF = +32·3, BD = −28·8, FD = +4·2, FA = +29·5, DA = −19·2,
   DE = −11·2.
8. BF = +30·7, BD = −34·4, FD = +5, FA = +31·1, DA = −21·1,
   DE = −11·0, AE = −17·4, AG = +28, EC = −26·9.
9. BF = +2·69, BD = −2·5, FD = +0·5, FA = +2·69, DA = −0·8,
   DE = −1·88.
10. BF = +4·7 = FA, BD = −4·4, FD = +1·0, FA = +4·7, DA = −1·6,
    DE = −3·1, AG = GC = +6·1, AE = −3·2, GE = +2·0, EC = −5·6.
11. BD = +4040 = DE, BK = −3750, DK = +500, KE = −800,
    KL = −3125, EL = +1250, EA = +3366, LA = +1630,
    LM = −2080, AM = 0.
12. BD = +11,070, BM = −10,500, DM = 0, DN = +1580,
    DE = +9500, MN = −10,500, EN = −500, EF = +7950,
    EO = +1800, NO = −9000, FO = −1000, FA = +6350,
    FG = +2150, OG = −7450, AG = −3000.
13. AD = +27·3, AF = −19·3, DF = −9·6, DC = +24,
    CF = −9·7, FG = −13·6.
14. AC = +8·5, AE = −6·0 = EF, CE = −3, CF = −4·2, CD = +9·0,
    FD = −1·4, FG = −8·0 = GB, DG = −7·0, DB = +11·3.
15. AD = −5·0, AC = +8·7, DC = +2·9, DF = −5·8 = −FC.
16. BC = −21·4, OC = −12·8, CA = −18·8.
17. AF = −10·0, AG = +8·7 = FG, FE = −15·0, GE = −8·7,
    GH = +17·3, ED = −27·5, EH = +13, HD = −11·5,
    HB = +26·0, DB = +17·5, DC = −40.
18. AB = −20·4, AC = +22·6, BC = +7·6, CD = +21·4.
19. AD = −11·5 = DE = EC, AC = +5·8, DC = +11·5 = EB,
    CB = +17·3, EF = −23.
20. AC = CB = 0, AD = +7·5, AH = −3·75, DE = +5·2, DH = −2·9,
    HE = +2·9, HK = −6·65.
21. AC = 1, CD = 0, AD = +5·8, DE = +4·1, AH = −2·9, HE = +2·3,
    HK = −5·2.
22. FA = +8, FG = 0, AB = +6·0, AG = −8·48, BG = 6·0, GH = −6·0,
    BC = +8·0, CH = +4·0.

**23.** DF = 0 = FG, DG = +15·9, DA = −11·25 = AG, GH = +11·25,
AH = +1·77, AB = −12·5 = BC, HB = −5·0, HK = +8·75,
HC = +5·3, KC = −8·75 = CE, KE = +12·4.

**24.** PM = −20, OM = +11·2, ML = −8·7 = LK, MN = −5·0,
ON = +15, LN = +5·0, NK = +10·0.

## Examples XXVI. (p. 190.)

**1.** 1·67″. **2.** 37·5 lbs. **4.** 10·64 lbs.
**3.** 40·2 tons, 42·35 tons, 21⅝ ft., 9⅛ ft. **8.** 66⅔ ft.
**5.** 62½ tons, 50 tons. **7.** 1760 lbs.-ft., 150 oz.
**9.** (1) 6, 6, 6·1, 6·2, 6·3, 6·5 tons; (2) 4° 48′, 9° 28′, 14° 2′, 18° 26′, 22° 37′.
**10.** 6·25, 6·29, 6·4, 6·5, 6·64 tons.

# CHAPTER VIII.

## Examples XXVII. (p. 197.)

**1.** (P ~ Q)$a$. **2.** 6·84 lbs.-ft. **3.** $3\cos(\theta - 50)$ lbs.-ft.
**4.** 251 lbs.-ft. **5.** 10 lbs.-ft. **8.** 2 lbs.-ft.
**9.** 30 grams-cms. **10.** ·43, 1·2, 1·95, 2, 1·5, ·78.
**11.** 8·62 oz.-inches. **12.** $10\sin\theta$ lbs.-ft. **15.** 149 lbs.-ft.

## Examples XXVIII. (p. 205.)

**1.** 150 lbs.-ft., 96 lbs.-ft. **2.** 2 tons-ft.
**3.** 80 lbs.-ft. At mid-point. **5.** (1) 0; (2) 125 lbs.-ft.
**6.** B.M. = 125 lbs.-ft., S.F. = 0; B.M. = 93·75 lbs.-ft., S.F. = 25 lbs.
**7.** $\frac{W(l-x)^2}{2l}$. **8.** Zero at the ends.
**9.** 30 lbs.-ft., S.F. = 5 lbs. At A and 4 ft. from A.
**10.** Max. B.M. 14·4 ft. from end = 207·4 tons-ft. Max. S.F. just before second support = −31·2 tons.
**11.** 3745 lbs. **12.** Max. 4·2 ft. from end of load = 21·4 tons-ft.
**13.** 400 lbs.-ft.
**14.** 1⅓ lbs. upwards at B, downwards at C; B.M. = 88⅓ lbs.-ft.; S.F. = 1⅓ lbs.
**15.** B.M. = 25⅓ lbs.-ft., S.F. = 4⅓ lbs. **16.** 1863 lbs.-ft., 7452 lbs.-ft.

## Examples. (p. 210.)

1. 876 lbs.-ft. 2. 1·85. 3. 16·8 tons-ft.

## Examples XXIX. (p. 215.)

1. B.M. = 3 tons-ft., S.F. = ½ ton. 2. B.M. = 25 cwt.-ft.
3. 180, 320, 420, 480, etc., lbs.-ft.
4. Max. 1000 lbs.-ft. at mid point. 5. Max. 25 tons-ft. at middle.
6. Max. $11\frac{2}{3}$ ft. from end. 7. Max. at A, B.M. = 52 tons-ft.
8. At A and 6·4 ft. from A. 9. B.M. = 3600 lbs.-ft., S.F. = 1000 lbs.
11. B.M. = 23 tons-ft., S.F. = $1\frac{1}{3}$ tons.
12. −600 at mid point of AB, +1200 at B, 0 at 57·7 ft. from A.

## Miscellaneous Examples.

### Chapters I.–VIII.

### O. (p. 217.)

1. At A, 4 tons; C, 16 tons; 12 tons; 12 tons.
2. 5 lbs., 36° 52′; 2·24 lbs., 26° 34′.
3. On diagonal 5·8″ from left edge. 4. 2·8 ft.

### P. (p. 218.)

1. P = ·71 lbs., W = 3·5 lbs.
5. X = 1·1, Y = 1·3, DC = −1·65, CA = +·86, CB = −1·6, AB = +2·2.

### Q. (p. 219.)

4. Pressures 13·8 and 8·2 lbs.

### R. (p. 220.)

1. Cuts AB at E, where BE = ⅓BA.
3. 3·42 lbs. 4. Width = $2(8 + 6\cos 30)$.

### S. (p. 222.)

1. 173·2 lbs. 2. 6 ft. from B.
4. At Q force upwards on PQ = $X_1$ and downwards on QR = $X_2$. Then $X_1 - X_2 = T$. Take moments for each part.

### T. (p. 223.)

1. P = 1·3, 55° to vertical; Q = 2·85, 7° to vertical.
2. 1·4 ft. 3. 20%, 40%, 50%.

## U. (p. 225.)

**1.** At B, 30 lbs.; at C, 90 lbs.; at D, 60 lbs. **2.** 17·4″

**3.** 10·8 lbs., 8·98 lbs. **4.** $\frac{3\mu W}{2\sin\theta + 3\mu\cos\theta}$.

**5.** AB = +18·75, AE = −22·5, BC = +25, BE = −31·25.

## V. (p. 226.)

**3.** $\frac{W}{P} = \cot(\phi + \alpha)$; $e = \cot(\phi+\alpha)\tan\alpha$. **4.** ·53 lb.

**5.** A = 250, B = 750, AD = −288, AB = +144, DB = +500, DC = −577, CB = −1154.

## W. (p. 227.)

**2.** From A to B $= \frac{Wx(BD)}{BC \,.\, AD}$, where AW $= x$.

From B to D $= \frac{Wx(AB)}{BC \,.\, AD}$, where DW $= x$. Max. at B = 15·6 stone.

**3.** 3·3 lbs. **4.** 24·5, 24·8, 21·7. **5.** 1·6.

## X. (p. 229.)

**1.** $1\frac{1}{3}$ tons and $6\frac{1}{3}$ tons. **3.** 24·7 lbs.

**4.** $W\cos\theta = 2w\cos\frac{\theta}{2}$; 35° 24′ to horizontal. **5.** 31·5 lbs

## Y. (p. 230.)

**2.** 1860 lbs. **3.** 3·8 lbs. **5.** 6″.

## Z. (p. 232)

**1.** 2·9. **2.** 9 nearly. **3.** 9·78. **4.** 4·47 lbs.

**5.** c. of g. of quadrilateral does not coincide with c. of g. of equal weights at the angular points.

# CHAPTER IX.

## Examples XXX. (p. 242.)

**1.** 1·7 lbs.

**2.** (1) Upper = 125 lbs.; lower, 195 lbs. (2) Upper, 312 lbs.; lower, 426 lbs.

**3.** 11·2 lbs., 14·3 lbs. **4.** 17·9 lbs., 17·9 lbs. horizontal.

**6.** $1\frac{1}{3}$ ft., $\frac{1}{2}$ lb. **7.** $\frac{1}{3}$BC from B.

**8.** $P=\mu W$, $R_1=\frac{W(a+\mu d)}{2a}$, $R_2=\frac{W(a-\mu d)}{2a}$.

**9.** Upper, 2 lbs.; lower, 4 lbs. **12.** 54·2 lbs.

**13.** $T=\frac{p}{5}$, $X=\frac{p}{3}+\frac{50}{3}$. **14.** $X=1{\cdot}38$, $Y=1{\cdot}0$.

**15.** $X=\cdot58$, $Y=2{\cdot}0$. At C, $X=\cdot58$.

**16.** $P=5{\cdot}77$. Top cylinder, 5·77. Second row, 11·5 and 5·77.

**18.** $X=\cdot18$, $Y=\cdot93$. **19.** $T=\frac{w}{4}\sec\beta$.

### Examples XXXI. (p. 249.)

**1.** 1·88 lbs. **2.** ·69 ton. **3.** 1·75 tons, 2·24 tons.

**4.** 330 lbs. **5.** $\frac{13}{18}$ lb. **6.** ·68 lb.

**8.** ·91, ·91, ·42 lb.

### Examples XXXII. (p. 252.)

**1.** BD = +8·7, BC = −5·2, AC = −6·1.

**2.** CD = +17·9, CE = −5·8, FE = −15.

**3.** AB = +16·8, EB = −8·3, ED = −8·6.

**4.** BE = −7·1, BD = −5·35, CD = +12·6.

**5.** AB = −·89, BC = +·67, CD = +·45.

**6.** HD = +·9, CD = −0·5.

### Examples XXXIII. (p. 256.)

**1.** R = 3·74 inclined at 79° to DC. Cuts DC produced 3·75″ from D and BC produced 9·2″ from C.

**2.** R = 10·63. Cuts FA 1·3″ from F and DC ·34″ from D.

**3.** B, 36·6; C, 91·9; E, 12·1; F, 68·4.

**4.** 7·55. Cuts BC ·86″ from B and AC produced 10″ from A.

**5.** 3·6. C = 7 lbs.-ft. **6.** 1·4, ·05″ below A.

**7.** R = 2·15, S = 2·78, T = 3·1.

**8.** R = 4·4. If $a$ is side, R cuts AB produced $1{\cdot}4a$ from B and ED produced $\cdot8a$ from D.

## CHAPTER X.

### Examples XXXIV. (p. 264.)

**2.** 39°. **3.** $\mu=\cdot13$. **9.** $\mu=\cdot58$.

**10.** $\mu=\cdot414$. **12.** $\mu=1{\cdot}33$. **13.** 58° 38′.

**17.** 1·9 ft., 1·07 ft. **19.** $\mu=0{\cdot}5$, P = 52 Kg.

**23.** $\tan\theta=\frac{7}{12}$, $\theta=23°\ 32'$. **26.** 10·1 lbs.

# CHAPTER XI.

## Examples XXXV. (p. 274.)

**1.** $4\frac{1}{12}''$ from left edge; 3·38″ from bottom edge.
**2.** $\bar{x}=7\cdot7$, $\bar{y}=14\cdot6$.
**3.** $\bar{x}=2\cdot4$, $\bar{y}=1\cdot04$.
**4.** $\bar{x}=4\cdot35$, $\bar{y}=3\cdot17$.
**5.** $\bar{x}=2\cdot27$, $\bar{y}=1\cdot74$.
**6.** 150,000 ft.-tons.
**7.** 0·14 ft.-ton.
**8.** 4500 ft.-tons.
**9.** 47,900 ft.-lbs.
**10.** $20\frac{2}{3}$ ft.-lbs.

## Examples XXXVI. (p. 281.)

**1.** $\frac{1}{8}r$ from common base.
**2.** 0·35 ft.
**3.** 2·57″ from centre.
**4.** ·164 from mid-point.
**5.** 32° 30′.
**7.** 11·46″ from centre.
**8.** $\frac{r}{3}$.
**9.** 32° 6′.
**10.** 306° 52′.
**11.** ·27″ from centre.
**12.** $\bar{x}=\cdot318$ ft., $\bar{y}=-0\cdot5$ ft.

# CHAPTER XII.

## Examples XXXVII. (p. 291.)

**1.** P = 10; components at E (0, 0, 11); F (0, 0, −1).
**2.** P = 10; E −32, −3, 0; F 32, −7, 80.
**3.** T = 21·2; A (10·6, $Y_1$, 15); B (0, $Y_2$, 0); $Y_1+Y_2=10\cdot6$.
**4.** 24·23, 4·22, 31·55.
**5.** R = 8; A (−4, $Y_2$, 0); B (12, $Y_1$, 2); $Y_1+Y_2=0$.
**6.** $Y_1=Y_2=35\cdot5$; $Y_3=78\cdot9$; $Y_1=Y_2=64\cdot4$; $Y_3=21\cdot1$.
**7.** T = 5·32; E (3·64, 5, 0); F (5·46, 5, 10).
**8.** T = 3·1; A (0, $Y_2$, 6); B (·8, $Y_1$, ·8); $Y_1+Y_2=0$.
**9.** P = 300; 106, 946, 0; 106, 466, 0.
**10.** 20·8 lbs.-ft.
**11.** −11·15, +3, +8·16.
**12.** 0, −40·2, $134\frac{1}{4}$; 0, 40 2, $201\frac{3}{4}$.
**14.** $\dfrac{W}{2\sqrt{\cot^2\alpha+\mu^2}}$.
**15.** $\dfrac{Pa\sqrt{3}}{2}$ or $\dfrac{Pa}{2}$.

## Miscellaneous Examples.

### I. (p. 294.)

**2.** 19·6 cm.; 372 grams.
**3.** 2 tons, $1\frac{2}{22}$ tons, $\frac{25}{28}$ ton.
**5.** R = ·1 ton; $\mu=0\cdot4$.

## II. (p. 295.)

1. $T_1 = \frac{W}{2}(\operatorname{cosec}\alpha + \tan\alpha)$; $T_2 = \frac{W}{2}(\tan\alpha - \operatorname{cosec}\alpha)$.
2. 50·7, 35·4, 73·9.
3. 4·36 tons, 20·1 tons.
4. 64·7 lbs.
5. CF = 1·87 lbs., FB = ·69 lb.

## III. (p. 296.)

1. 48½, 29, 26·5 lbs.
2. ·29 lb.
4. Upper locked, $\tan\alpha = \frac{\mu a}{a+b+\mu h}$; lower locked, $\tan\beta = \frac{\mu b}{a+b-\mu h}$.

## IV. (p. 298.)

1. 578 lbs.
2. A, $\frac{W}{3}$; B, $\frac{W}{6}$; C, $\frac{W}{6}$; D, $\frac{W}{3}$.
3. AB + 4·6, BC = +2·3, AC = +5·8, CD = +4·3; at A 11·6, at D 6·6.
4. 53 lbs.
5. $\mu = \frac{1}{3}$, $\frac{3}{8}$ ton.

## V. (p. 299.)

1. ·73 lb.
3. $\frac{(W+W')\cot\alpha}{4}$.

## VI. (p. 301.)

1. 0·38.
2. 8½, 3½ tons
4. C, $\frac{2w}{\tan\alpha+\tan\beta}$ and $\frac{w(\tan\beta-\tan\theta)}{\tan\alpha+\tan\beta}$.
   D, $\frac{2w}{\tan\alpha+\tan\beta}$, $\frac{w(\tan\alpha+\tan\theta)}{\tan\alpha+\tan\beta}$.
5. 3·1, 2·1.

## VII. (p. 302.)

1. (1) W cosec 30. (2) 2W cosec 30; W cot 30, W.
4. P = 3·0 lbs., Q = 1·74 lbs., R = 3·43 lbs.

## VIII. (p. 304.)

1. $\tan\alpha = \frac{W}{T_0}$, $\tan\beta = \frac{2W}{T_0}$, etc.
3. 3·25.
4. $\lambda = 15°$. If K be mid-point of string, suppose AK to be fixed, then C′ lies on a segment drawn on AK with vertical angle 75°. Also BC′ = 5½. C′ is 2 ft. from A.